Symmetries of energy flow operators

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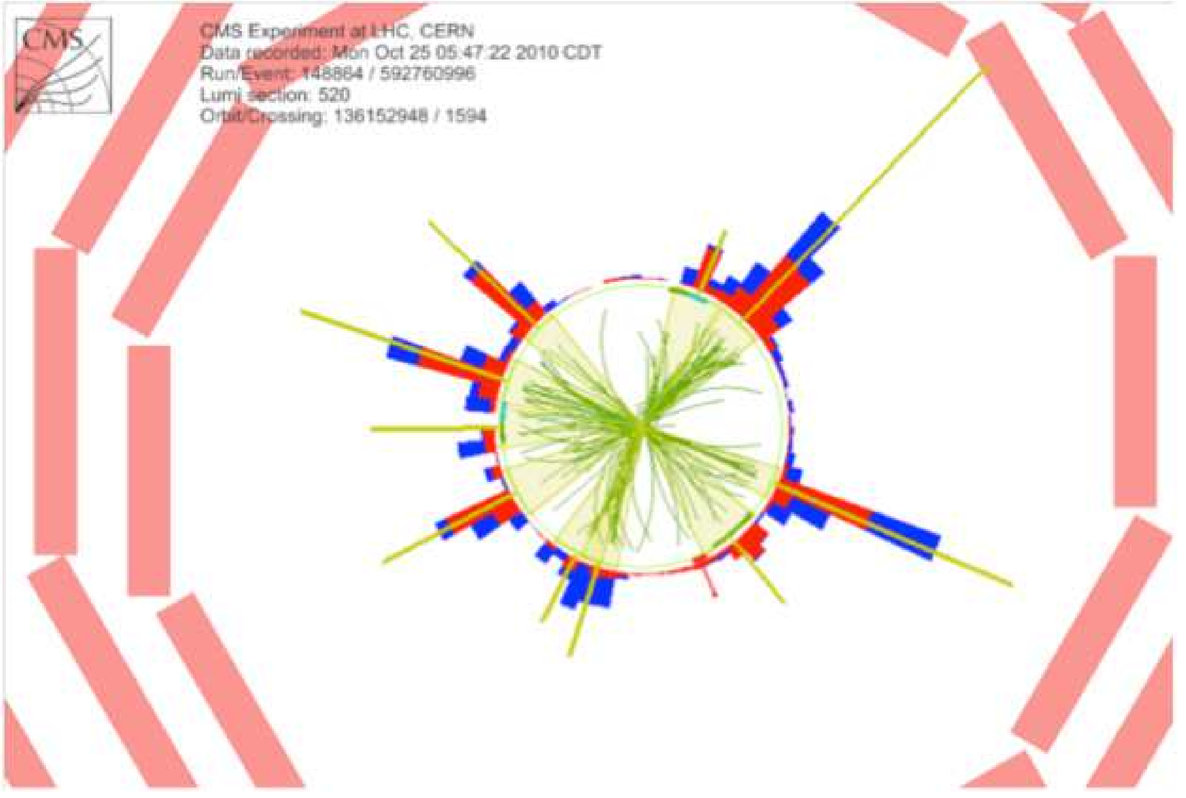
Final states at LHC:



- ✔ A lot of particles produced
- ✔ Energy of particles is deposit at the calorimeters
- ✔ Admit description in terms of the energy distribution on the celestial sphere

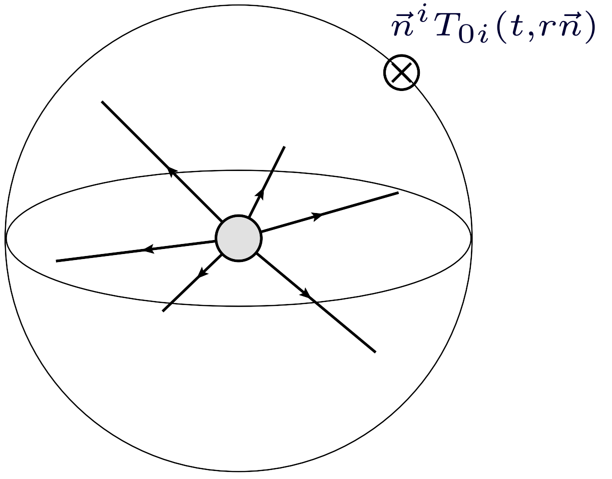
Energy flow operators

Angular distribution of the energy

$$f(\vec{n}_1,\dots,\vec{n}_L)=\sum_X |\langle in|X\rangle|^2 E_1\delta^{(2)}(\Omega_{\vec{p}_1}-\Omega_{\vec{n}_1})\dots E_L\delta^{(2)}(\Omega_{\vec{p}_L}-\Omega_{\vec{n}_L})$$

Particles with momenta $p_i=(E_i,\vec{p}_i)$ enter calorimeters located in the direction $\vec{n}_i$

Energy flow operator [Sveshnikov,Tkachov]



$$\mathcal{E}(\vec{n})|X\rangle=\sum_a E_a\,\delta^{(2)}(\Omega_{\vec{p}_a}-\Omega_{\vec{n}})|X\rangle$$

Expression in terms of the stress energy tensor

$$\mathcal{E}(\vec{n})=\lim_{r\to\infty}\int_0^\infty dt\,r^2\,\vec{n}^i T_{0i}(t,r\vec{n})$$

Correlation functions of the flow operators [GK,Sterman],[Belitsky,GK,Sterman],[Hofman,Maldacena]

$$f(\vec{n}_1,\dots,\vec{n}_L)=\langle in|\mathcal{E}(\vec{n}_1)\dots\mathcal{E}(\vec{n}_L)|in\rangle$$

A powerful approach to computing QCD observables [Chen,Moult,Zhu]

What are symmetries of the energy flow operators?

Covariant definition and generalization

Energy flow operator in $d-$dimensions

$$\mathcal{E}(n) = \lim_{r\to\infty} r^{d-2} \int_{-\infty}^{\infty} dt\, T_{\mu_1\mu_2}(rn + t\bar{n})\bar{n}^{\mu_1}\bar{n}^{\mu_2}$$

$n = (1, \vec{n})$ points towards the calorimeter, $\bar{n}$ auxiliary null vector, $(n\bar{n}) = 1$.

Introduce energy resolution ω

$$\mathcal{E}_\omega(n) = \lim_{r\to\infty} r^{d-2} \int_{-\infty}^{\infty} dt\ \mathrm{e}^{-it\omega}\ T_{\mu_1\mu_2}(rn + t\bar{n})\bar{n}^{\mu_1} \dots \bar{n}^{\mu_S}$$

Generalization to higher spin conserved currents $J_{\mu_1\dots\mu_S}(x)$

$$\mathcal{J}_{\omega,S}(n) = \lim_{r\to\infty} r^{d-2} \int_{-\infty}^{\infty} dt\, \mathrm{e}^{-it\omega}\ J_{\mu_1\dots\mu_S}(rn + t\bar{n})\bar{n}^{\mu_1} \dots \bar{n}^{\mu_S}$$

Global charges

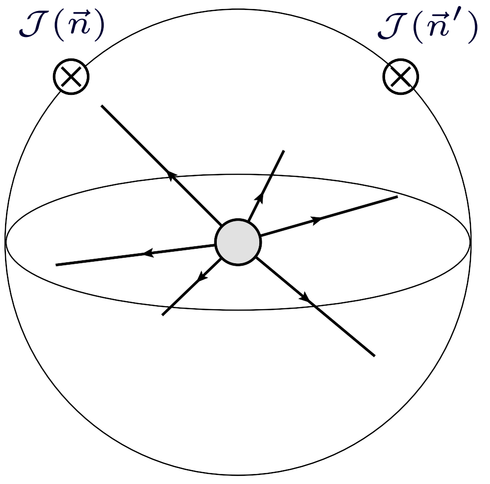
$$\int d^{d-2}n\, \mathcal{E}_\omega(n) = \mathbb{P}_- - i\omega\, \mathbb{M}_{+-} + O(\omega^2)$$

$\mathbb{P}_-$ total momentum, $\mathbb{M}_{+-}$ Lorentz boost; satisfy Poincaré algebra

The goal is to establish the algebra of the flow operators $\mathcal{J}_{\omega,S}(n)$

What we expect

The detectors are located on the celestial sphere $n = (1, \vec{n})$



The flow operators are space-like separated for $n \neq n'$

Their commutator should be localized at $n = n'$

$$[\mathcal{J}_{\omega,S}(n), \mathcal{J}_{\omega',S'}(n')] \sim \delta^{(d-2)}(n, n')$$

Which operators appear on the right-hand side?

For $n = n'$ the effective dynamics takes place on the light-cone

Relation to low dimensional symmetries?

Warm up example

In $d = 2$ dimensions

$$\mathcal{E}_\omega = \lim_{r\to\infty} \int_{-\infty}^{\infty} dt\ \mathrm{e}^{-it\omega}\ T_{--}(rn + t\bar{n})$$

Commutation relations of $T_{--} = T_{\mu_1\mu_2}\bar{n}^{\mu_1}\bar{n}^{\mu_2}$

$$[T_{--}(x_1^-), T_{--}(x_2^-)] = i(\partial_{1-} - \partial_{2-})\Big(T_{--}(x_1^-)\delta(x_1^- - x_2^-)\Big) - \frac{ic}{24\pi}\delta'''(x_1^- - x_2^-)$$

The algebra of the flow operators in $d = 2$ dimension

$$[\mathcal{E}_\omega, \mathcal{E}_{\omega'}] = (\omega' - \omega)\mathcal{E}_{\omega+\omega'} - \frac{c}{12}\omega^3\delta(\omega + \omega')$$

Formal expansion

$$\mathcal{E}_\omega = \sum_k \frac{(i\omega)^k}{k!} L_k\ ,$$

Generalized ANEC operators

Virasoro algebra

$$[L_m, L_k] = (m - k)L_{m+k} + \frac{c}{12}m(m^2 - 1)\delta_{m+k,0}$$

The energy flow operator is a generating function of the Virasoro modes

Warm up example II

Extention to higher spin currents

$$\mathcal{J}_{\omega,S} = \lim_{r\to\infty} \int_{-\infty}^{\infty} dt\, \mathrm{e}^{-it\omega}\;\; J_{\mu_1\ldots\mu_S}(rn + t\bar{n})\bar{n}^{\mu_1}\ldots\bar{n}^{\mu_S}$$

Generalized Zamolodchikov algebra

$$[J_{S_1}(x_1^-), J_{S_2}(x_2^-)] = -\sum_{S_3=S_{\mathrm{min}}}^{S_1+S_2-2} f_{S_1 S_2}^{S_3}(i\partial_1, i\partial_2)\Big(J_{S_3}(x_1^-)\delta(x_1^- - x_2^-)\Big) - i\delta_{S_1 S_2}\frac{\tilde{c}_{S_1}}{24\pi}\delta^{(2S_1+1)}(x_1^- - x_2^-)$$

Algebra of higher spin flow operators

$$[\mathcal{J}_{\omega,S}, \mathcal{J}_{\omega',S'}] = \sum_{S''=S_{\mathrm{min}}}^{S+S'-2} f_{SS'}^{S''}(\omega,\omega')\mathcal{J}_{\omega+\omega',S''} - \delta_{SS'}\delta(\omega+\omega')\omega^{2S+1}\frac{\tilde{c}_S}{12}$$

Formal expansion

$$\mathcal{J}_{\omega,S} = \sum_k \frac{(i\omega)^k}{k!} V_k^{i=S-2}$$

W_∞ algebra [Pope,Romans,Shen]

$$[V_m^i, V_n^j] = \sum_{\ell\geq 0} g_{2\ell}^{ij}(m,n)V_{m+n}^{i+j-2\ell} + c_i(m)\delta^{ij}\delta_{m+n,0}$$

Leading twist operators in d dimensions

Conserved currents in a free $U(1)$ gauge theory (fermions + photons)

$$J_S^{(\psi)} \sim \bar{\psi}\gamma_-(i\partial_-)^{S-1}\psi + \dots$$

$$J_S^{(g)} \sim F_{-\mu}(i\partial_-)^{S-2}F_-{}^{\mu} + \dots$$

All Lorentz index are contracted with $\bar{n}^\mu$

Conformal symmetry fixes form of the operators built out of $\Phi = \{\psi\,, F_{-\alpha}\}$

$$J_S(x) = \bar{\Phi}(x) P_S(i\overrightarrow{\partial}_-, i\overleftarrow{\partial}_-)\Phi(x)$$

$$P_S(p_1,p_2) = (p_1+p_2)^S C_S^{2j-\frac{1}{2}}\left(\frac{p_1-p_2}{p_1+p_2}\right) \qquad \textit{Gegenbauer polynomial}$$

Conformal spins $j_\psi = d/4, \quad j_g = (d+2)/4$

Stress-energy tensor $T_{--} = \frac{1}{2(d-1)} J_{S=2}^{(\psi)} + J_{S=2}^{(g)}$

Generalized energy flow operator

$$\mathcal{J}_{\omega,S}(n) = \lim_{r\to\infty} r^{d-2}\int_{-\infty}^{\infty} dt\, \mathrm{e}^{-it\omega}\ J_S(rn+t\bar{n})$$

Commutation relations

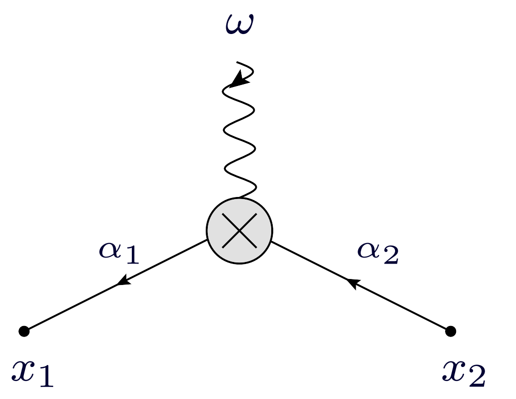
$$[\mathcal{J}_{\omega,S}(n), \mathcal{J}_{\omega',S'}(n')] \overset{?}{\sim} \mathcal{J}_{\omega'',S''}(n'')$$

Insert inside the correlation function

$$\langle\Phi(x_1)\mathcal{J}_{\omega,S}(n)\mathcal{J}_{\omega',S'}(n')\bar{\Phi}(x_2)\rangle - \langle\Phi(x_1)\mathcal{J}_{\omega',S'}(n')\mathcal{J}_{\omega,S}(n)\bar{\Phi}(x_2)\rangle \sim \langle\Phi(x_1)\mathcal{J}_{\omega'',S''}(n)\bar{\Phi}(x_2)\rangle$$

Three-point function

$$\langle\Phi(x_1)\mathcal{J}_{\omega,S}(n)\bar{\Phi}(x_2)\rangle =$$



Transition amplitude: $x_2 \to$ *Detector* $\to x_1$

The particle enters the detector with the energy α_2 and leaves it with the energy $\alpha_1 = \alpha_2 + \omega$

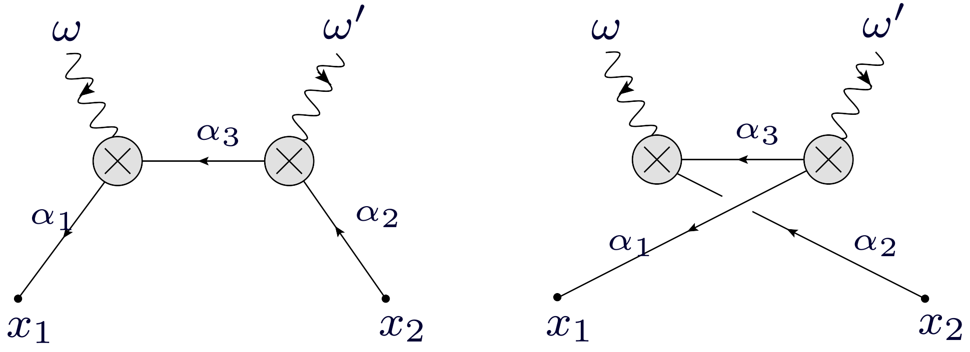
$$\langle\Phi(x_1)\mathcal{J}_{\omega,S}(n)\bar{\Phi}(x_2)\rangle \sim \int_0^\infty d\alpha_1\,(\alpha_1(\alpha_1-\omega))^{2j-1}\theta(\alpha_1-\omega)\,\mathrm{e}^{-i\omega(nx_2)-i\alpha_1(nx_{12})}\,P_S(\alpha_1,-\alpha_1+\omega)$$

P_S is the energy dependent weight

Commutation relations II

Four-point function (connected part)

$$\langle\Phi(x_1)\mathcal{J}_{\omega,S}(n)\mathcal{J}_{\omega',S'}(n')\bar{\Phi}(x_2)\rangle =$$



Particle goes subsequently through the two detectors

$$\langle\Phi(x_1)[\mathcal{J}_{\omega,S}(n),\mathcal{J}_{\omega',S'}(n')]\bar{\Phi}(x_2)\rangle_c \sim \delta^{(d-2)}(n,n')\,\mathrm{e}^{-i\omega''(nx_2)}$$
$$\times\int_0^\infty d\alpha_1\,(\alpha_1(\alpha_1-\omega''))^{2j-1}\theta(\alpha_1-\omega'')\,\mathrm{e}^{-i\alpha_1(nx_{12})}\,Q(\alpha_1)$$

Total energy $\omega''=\omega+\omega'$, polynomial $Q(\alpha_1)$ is bilinear in P's

$$Q(\alpha_1) =(\alpha_1-\omega)^{2s-1}P_S(\alpha_1,-\alpha_1+\omega)P_{S'}(\alpha_1-\omega,-\alpha_1+\omega'')$$
$$-(\alpha_1-\omega')^{2s-1}P_{S'}(\alpha_1,-\alpha_1+\omega')P_S(\alpha_1-\omega',-\alpha_1+\omega'')$$

j / s are conformal / Lorentz spin of the field

Algebra of the energy flow operators

Expansion over orthogonal (Gegenbauer) polynomials

$$Q(\alpha_1) = \sum_{S''=2s}^{S+S'-2} C_{SS'}^{S''}(\omega,\omega') P_{S''}(\alpha_1, -\alpha_1+\omega'')$$

Algebra of the flow operators

$$[\mathcal{J}_{\omega,S}(n), \mathcal{J}_{\omega',S'}(n')] = \delta^{(d-2)}(n,n') \sum_{S''=2s}^{S+S'-2} C_{SS'}^{S''}(\omega,\omega') \mathcal{J}_{\omega+\omega',S''}(n) - \delta_{SS'}\delta(\omega+\omega')\omega^{2S-1}\Omega_S\big(n,n'\big)$$

The central charge comes from a disconnected part of 4pt function

$$\Omega_S(n,n') \sim \langle \mathcal{J}_{\omega,S}(n)\mathcal{J}_{\omega',S'}(n')\rangle \sim [\delta^{(d-2)}(n,n')]^2$$

It is finite for $d=2$ and diverges at $d>2$ (= Vol_{d-2})

Structure constants

$$C_{SS'}^{S''}(\omega,\omega') = \frac{1+(-1)^{S+S'-S''}}{2} \sum_{k=0}^{S+S'-S''-1} f_k\, \omega^k (\omega')^{S+S'-S''-1-k}$$

Satisfy Jacobi identity, vanish for odd $S+S'-S''$

Virasoro and W algebra

Spin-2 flow operators $\mathcal{E}_\omega(n) = \mathcal{J}_{\omega,S=2}(n)$

$$[\mathcal{E}_\omega(n), \mathcal{E}_{\omega'}(n')] = \delta^{(d-2)}(n,n')(\omega' - \omega)\mathcal{E}_{\omega+\omega'}(n) - \omega^3\delta(\omega+\omega')c_d(n,n')$$

Take a universal form for any d

Virasoro algebra in d dimensions [Casini,Teste,Torrona]

$$[L_m, L_k] = \delta^{(d-2)}(n,n')(m-k)L_{m+k} + \frac{c_d(n,n')}{12}m(m^2-1)\delta_{m+k,0}$$

Higher spin flow operators

$$[\mathcal{E}_\omega(n), \mathcal{J}_{\omega',S'}(n')] = -\delta^{(d-2)}(n,n')((S'-1)\omega - \omega')\mathcal{J}_{\omega+\omega',S'}(n') + \text{[lower spins]}$$

$$[\mathcal{J}_{\omega,S}(n), \mathcal{J}_{\omega',S'}(n')] = \delta^{(d-2)}(n,n')\sum_{S''=2s}^{S+S'-2} C_{SS'}^{S''}(\omega,\omega')\mathcal{J}_{\omega+\omega',S''}(n) - \delta_{SS'}\delta(\omega+\omega')\omega^{2S-1}\Omega_S(n,n')$$

W algebra in d dimensions

$$[V_m^i, V_n^j] = \delta^{(d-2)}(n,n')\sum_{\ell\geq 0} f_{2\ell}^{ij}(m,n)V_{m+n}^{i+j-2\ell} + c_m(n,n')\delta^{ij}\delta_{m+n,0}$$

The structure constants depend on d

From $d=2$ to $d=4$

The structure constants of the algebra

$$C_{SS'}^{(\Phi),S''}(\omega,\omega') \sim \int_0^1 dt\,(t(1-t))^{2j-1}(t-\varepsilon)^{2s-1}\,P_S(t,\varepsilon-t)P_{S'}(t-\varepsilon,1-t)P_{S''}(t,1-t)\Big|_{\epsilon=\frac{\omega}{\omega+\omega'}}$$

Depend on d through conformal spin of fields $j_\Phi = (\Delta_\Phi + s_\Phi)/2$ and their Lorentz spin s_Φ

Scalars in $d=4$ versus fermions in $d=2$

$$j_\phi(d=4) = j_\psi(d=2) = \frac{1}{2}$$

The structure constants in $d=2$

$$C_{SS'}^{(\psi),S''}\Big|_{d=2} = W_{1+\infty} \text{ algebra}$$

Intriging relation between the structure constants in $d=2$ and $d=4$

$$C_{SS'}^{(\psi),S''}\Big|_{d=2} = \left[a_+\,\mathrm{e}^{\partial_S} + a_-\,\mathrm{e}^{-\partial_S} + b_+\,\mathrm{e}^{\partial_{S''}} + b_-\,\mathrm{e}^{-\partial_{S''}}\right] C_{SS'}^{(\phi),S''}\Big|_{d=4}$$

Involves a finite-difference operator

Conclusions and open questions

- ✔ The flow operators are interesting objects in a gauge theory
- ✔ They form a closed algebra in a free gauge theory
- ✔ For $d = 2$ this algebra coincides with the $W-$algebra in CFT$_2$
- ✔ In $d = 4$ dimensions the structure constants are related to those of the $W-$algebra by a linear finite-difference operator
- ✔ What happens with the algebra in an interacting theory?

  The $U(1)$ current and the stress-energy tensor are protected, high spin currents become anomalous