E_{11} exceptional field theory

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Joint work with Guillaume Bossard and Ergin Sezgin

[2103.13411] [1907.02080 JHEP 1910 (2019) 165]

Also in addition with Jakob Palmkvist and Chris Pope

[1703.01305 JHEP 1705 (2017) 020]

Context

- Toroidal reduction of $D = 11$ supergravity on T^n **[Cremmer Julia]** $\Rightarrow$ max. SUGRA in $D = 11-n$ dimensions with global E_n
- Symmetry acts on scalars non-linearly and p-forms linearly: E_n tensor hierarchy **[de Wit, Nicolai Samtleben]**

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$$M_{11} = M_{11-n} \times T^n \qquad \text{coordinates} \quad (x^\mu, y^m)$$

$$\delta_\xi g_{mn}(x,\, y\,) = L_\xi g_{mn} = \xi^p \partial_p g_{mn} + \partial_m \xi^p g_{pn} + \partial_n \xi^p g_{mp}$$

for ξ^p along T^n.

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T^n red. $\quad$ ↑ scalar on M_{11-n}

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More of E_n from local matter gauge trm. in $D = 11$

- But $\exists$ also truly hidden E_n transformations. Require specific Chern–Simons term. Important for U-duality...

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For $\mathfrak{e}_n$-valued parameter $\partial_\bullet \xi^\bullet$
need to extend space since
E_n cannot act on torus y^m!
Replace $y^m \to Y^M \in R_1$
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	R_1	R_2
E_6	$\mathbf{27}$	$\overline{\mathbf{27}}$
E_7	$\mathbf{56}$	$\mathbf{133} \oplus \mathbf{1}$
E_8	$\mathbf{248}$	$\mathbf{3875} \oplus \mathbf{248} \oplus \mathbf{1}$

Exceptional field theory (ExFT)

Important point: Gauge transformations $\delta_\xi \mathcal{M} = \mathcal{L}_\xi \mathcal{M}$ only close when section constraint is imposed (NB $n \leq 7$)

$$\partial_P \otimes \partial_Q \Big|_{R_2} = 0$$

[Coimbra, Waldram Strickland-Constable] [Berman Perry]

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Include other fields ($g_{\mu\nu}$, A_μ^M,$\ldots$) from E_n tensor hierarchy and x^μ diffeos to obtain E_n ExFT

[Hohm Samtleben] $(n \leq 8)$

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Include other fields ($g_{\mu\nu}$, A_μ^M,...) from E_n tensor hierarchy and x^μ diffeos to obtain E_n ExFT [Hohm Samtleben] $(n \leq 8)$

- Uniquely fixed by symmetries. Contains $D = 11$ and IIB
- For $n = 8$ need ancillary gauge parameter for closure of gen. diffeo. Related to extra constrained fields
- For $n = 9$ these constrained fields are intertwined indecomposably with tensor hierarchy fields [Bossard, Ciceri Inverso, AK, Samtleben]

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Our work: Construct ExFT for E_{11}

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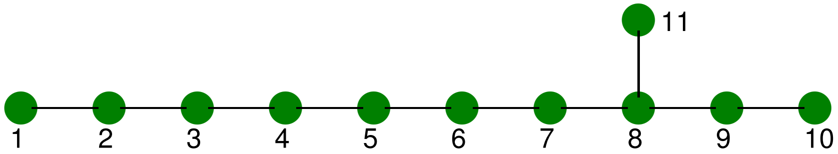
Results

- Pseudo-Lagrangian and (twisted) duality equation, invariant under E_{11} generalised diffeomorphisms
- Reduces to non-linear $D = 11$ SUGRA and ExFT

[Need many new E_{11} identities. Most proved, some only partially]

Some facts about E_{11}

∞-dim'l Kac–Moody algebra
Complete list of generators/
structure constants unknown



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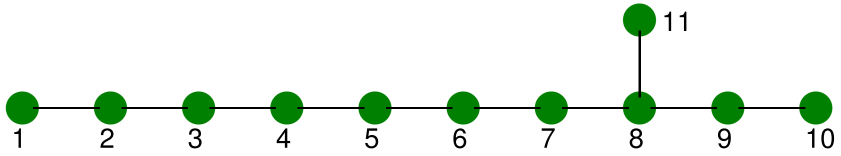

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Possible to define highest weight representations $R(\Lambda)$ [Kac]

Conjugate lowest weight $\overline{R(\Lambda)}$

hst. weight, comb. of fund. weights Λ_i

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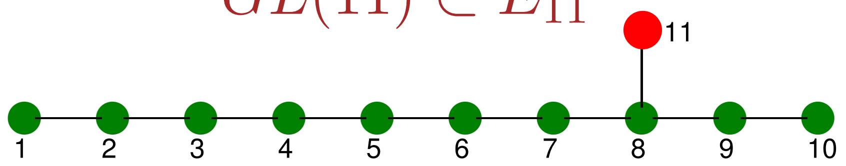

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Useful to consider graded decompositions [West] [Fischbacher Nicolai]

adjoint $\mathfrak{e}_{11}$: $\ldots, F_{n_1n_2n_3}, K^m{}_n, E^{n_1n_2n_3}, E^{n_1\ldots n_6}, E^{n_1\ldots n_8,n_9}, \ldots$

$R(\Lambda_1)$: $\ldots, P_{n_1\ldots n_5}, P_{n_1n_2}, P^m$

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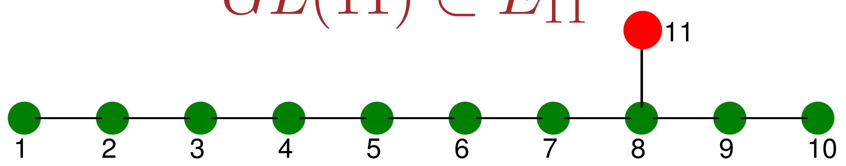
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$m, n = 0, 1, \dots, 10$

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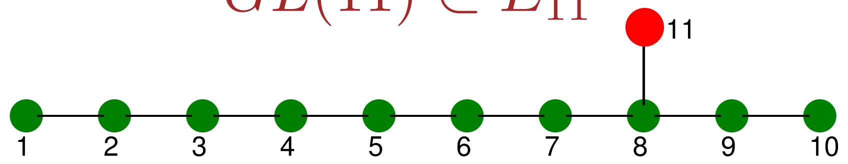

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$R(\Lambda_1)$: $\ldots, P_{n_1\ldots n_5}, P_{n_1n_2}, P^m \longleftarrow$ $D = 11$ coords

$\ell{=}-\frac{7}{2}$ $\ell{=}-\frac{5}{2}$ $\ell{=}-\frac{3}{2}$ (other: 'brane coords')

Ingredients of E_{11} ExFT

temp. involution
$\exists$ in $R(\Lambda_1)$

Following [West] take the coordinates z^M of the extended space in E_{11} rep. $R_1{=}R(\Lambda_1)$. Generalised metric $\mathcal{M} = \mathcal{V}^\dagger \eta \mathcal{V}$

$$\mathcal{M}(z) \to g^\dagger \mathcal{M}(g^{-1}z)g$$

'non-linear realisation of $E_{11} \ltimes \ell_1$'

under rigid E_{11}.

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Useful to write $\mathfrak{e}_{11}$ in $R(\Lambda_1)$ representation:

$$t^\alpha \mapsto T^{\alpha M}{}_N\,, \quad \mathcal{M} \mapsto \mathcal{M}_{MN}\,, \quad \mathcal{M}^{PS}\partial_M \mathcal{M}_{SQ} = J_{M\alpha}T^{\alpha P}{}_Q$$

Section constraint

$$T^{\alpha P}{}_M T_\alpha{}^Q{}_N \partial_P \otimes \partial_Q = -\frac{1}{2}\partial_M \otimes \partial_N + \partial_N \otimes \partial_M$$

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Instead: Use ExFT methods and extra fields for gauge-invariance

today

Tensor hierarchy extension

For any $\mathfrak{e}_n$ tensor hierarchy algebra $\mathcal{T}(\mathfrak{e}_n)$ encodes ExFT fields. Graded Lie superalgebra [Palmkvist]

$$\mathcal{T}(\mathfrak{e}_n) = \bigoplus_{p\in\mathbb{Z}} \mathcal{T}_p(\mathfrak{e}_n) \qquad \left[\mathcal{T}_p \cong \mathcal{T}^*_{9-n-p}\right]$$

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For $\mathfrak{e}_{11}$: existence of $\mathcal{T} \equiv \mathcal{T}(\mathfrak{e}_{11})$ proved in [1703.01305], structure

$$\mathcal{T}_0 = \underbrace{\left[\mathfrak{e}_{11} \oplus (R(\Lambda_2) \oplus \ldots)\right]}_{\widehat{\text{adj}}} \oplus \underbrace{\left[R(\Lambda_{10}) \oplus \ldots\right]}_{D_0}$$

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indecomposable sum
of $\mathfrak{e}_{11}$ representations

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

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$$\mathcal{T}(\mathfrak{e}_n) = \bigoplus_{p\in\mathbb{Z}} \mathcal{T}_p(\mathfrak{e}_n) \qquad \left[\mathcal{T}_p \cong \mathcal{T}^*_{9-n-p}\right]$$

$\mathcal{T}_p(\mathfrak{e}_n)$ contains the p-forms; $\mathbb{Z}_2$-even/odd depending on p

For $\mathfrak{e}_{11}$: existence of $\mathcal{T} \equiv \mathcal{T}(\mathfrak{e}_{11})$ proved in [1703.01305], structure

$$\mathcal{T}_0 = \underbrace{\left[\mathfrak{e}_{11} \oplus \left(R(\Lambda_2) \oplus \ldots\right)\right]}_{\widehat{\text{adj}}} \oplus \underbrace{\left[R(\Lambda_{10}) \oplus \ldots\right]}_{D_0}$$

indecomposable sum of $\mathfrak{e}_{11}$ representations

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

write as $t^{\widehat{\alpha}} = (t^{\alpha}, t^{\tilde{\alpha}})$ $\quad [t^{\alpha}, t^{\tilde{\alpha}}] = -T^{\alpha\tilde{\alpha}}{}_{\tilde{\beta}} t^{\tilde{\beta}} - K^{\alpha\tilde{\alpha}}{}_{\beta} t^{\beta}$

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something entangled with E_{11}!

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Positive levels $\mathcal{T}_{p>0}$ are sums of highest weights, e.g.

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In $GL(11)$ decomposition

$$t_I \in \mathcal{T}_{-1}: \quad \ldots, K^{n_1 n_2}{}_m,\ K^{n_1 n_2 n_3 n_4},\ K^{n_1 \ldots n_7},\ K^{n_1 \ldots n_9;m}, \ldots$$

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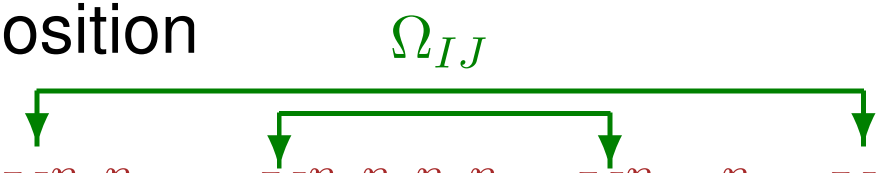

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⇒ candidate E_{11}-covariant duality equation $\mathcal{M}_{IJ} = (\mathcal{V}^\dagger\eta\mathcal{V})_{IJ}$

$$\mathcal{M}_{IJ}F^J = \Omega_{IJ}F^J$$

but what is F^I??

E_{11} field strengths

Would like F^I to contain the $\mathfrak{e}_{11}$ current components

$J_M{}^\alpha = \kappa^{\alpha\beta} J_{M\beta}$ $\longrightarrow$ need some tensor with indices I, M, α

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not E_{11} tensor!

Indecomposable

Get some E_{11}-invariant tensors from $\mathcal{T}$, e.g.

$$[P^M, \bar{t}_{\widehat{\alpha}}] = C^{IM}{}_{\widehat{\alpha}} t_I \,, \qquad C^{IM}{}_{\widehat{\alpha}} = (C^{IM}{}_\alpha, C^{IM}{}_{\tilde{\alpha}})$$

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$$F^I = C^{IM}{}_\alpha J_M{}^\alpha + C^{IM}{}_{\tilde{\alpha}} \chi_M{}^{\tilde{\alpha}} + C^{IM}{}_{\widehat{\Lambda}} \zeta_M{}^{\widehat{\Lambda}}$$

For gauge-invariance of duality equation need more fields

$$\begin{array}{lllll} \text{index} & M & & \tilde{\alpha} & \widehat{\Lambda} \\ & R(\Lambda_1) \otimes R(\Lambda_1) = & R(2\Lambda_1) \oplus & R(\Lambda_2) \oplus & R_{\text{section}} \end{array}$$

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$$\overset{M}{R(\Lambda_1)} \otimes R(\Lambda_1) = R(2\Lambda_1) \oplus \underset{\Pi^{\tilde\alpha}{}_{MN}}{\overset{\tilde\alpha}{R(\Lambda_2)}} \oplus \underset{\Pi^{\hat\Lambda}{}_{MN}}{\overset{\hat\Lambda}{R_{\text{section}}}}$$

E_{11} gauge transformations

$$\delta_\xi J_M{}^\alpha = \mathcal{L}_\xi J_M{}^\alpha + T^{\alpha N}{}_P \big(\partial_M \partial_N \xi^P + \mathcal{M}_{NQ}\mathcal{M}^{PR}\partial_R\partial_M\xi^Q\big)$$

$$\delta_\xi \chi_M{}^{\tilde\alpha} = \mathcal{L}_\xi \chi_M{}^{\tilde\alpha} + T^{\tilde\alpha N}{}_P \big(\partial_M \partial_N \xi^P + \mathcal{M}_{NQ}\mathcal{M}^{PR}\partial_R\partial_M\xi^Q\big) + \Pi^{\tilde\alpha}{}_{QP}\mathcal{M}^{NQ}\partial_M\partial_Q\xi^P$$

$$\delta_\xi \zeta_M{}^{\widehat\Lambda} = \mathcal{L}_\xi \zeta_M{}^{\widehat\Lambda} + \Pi^{\widehat\Lambda}{}_{QP}\mathcal{M}^{NQ}\partial_M\partial_Q\xi^P$$

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&\qquad + \Pi^{\tilde\alpha}{}_{QP}\mathcal{M}^{NQ}\partial_M\partial_Q\xi^P\\
\delta_\xi \zeta_M{}^{\widehat\Lambda} &= \mathcal{L}_\xi \zeta_M{}^{\widehat\Lambda} + \Pi^{\widehat\Lambda}{}_{QP}\mathcal{M}^{NQ}\partial_M\partial_Q\xi^P
\end{aligned}$$

give gauge-invariant duality equation

$$\mathcal{M}_{IJ}F^J = \Omega_{IJ}F^J$$

[1907.02080]

if ‘master identity’ satisfied

$$\Omega_{IJ}C^{JM}{}_{\hat\alpha}T^{\hat\alpha N}{}_Q = \overline{C}_{IQ}{}^{\tilde\beta}\Pi_{\tilde\beta}{}^{MN} + \overline{C}_{IQ}{}^{\widehat\Lambda}\Pi_{\widehat\Lambda}{}^{MN}$$

indices moved with η

Only partial proof of this identity available!

Constrained fields

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Expect (pseudo-)Lagrangian of ExFT type [Hohm Samtleben]

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Expect (pseudo-)Lagrangian of ExFT type [Hohm Samtleben]

Technical point: Recall that for E_n ExFT with $n \geq 8$ new structures appear due to non-closure of generalised diffeomorphisms [Coimbra, Waldram Strickland–Constable] [Berman, Cederwall AK, Thompson] [Hohm Samtleben] [Cederwall Palmkvist] .

Requires 'ancillary' gauge parameter $\Sigma_M{}^{\tilde{I}}$ where $\tilde{I}$ labels E_{11} representation $R(\Lambda_3) \oplus \ldots$, index M section constrained.

Have invariant tensor $C^{\tilde{I}}{}_{P\widehat{\alpha}}$

E_{11} ExFT pseudo-Lagrangian

Write in terms of four pieces

$$\mathcal{L}_{E_{11}} = \mathcal{L}_{\text{pot}_1} + \mathcal{L}_{\text{pot}_2} + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{top}}$$

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'Universal potential term' [Hohm Samtleben] [Cederwall Palmkvist] only E_{11} current

$$\mathcal{L}_{\text{pot}_1} = -\frac{1}{4}\kappa_{\alpha\beta}\mathcal{M}^{MN} J_M{}^\alpha J_N{}^\beta + \frac{1}{2} J_{M\alpha} T^{\beta M}{}_P \mathcal{M}^{PQ} T^{\alpha N}{}_Q J_{N\beta}$$

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'Ancillary potential term' **[Hohm Samtleben]** **[Cederwall Palmkvist]**

E_{11} current
and constrained $\chi_M{}^{\tilde{\alpha}}$

$$\mathcal{L}_{\text{pot}_2} = -\frac{1}{2}\mathcal{M}_{\tilde{I}\tilde{J}} C^{\tilde{I}}{}_{P\hat{\alpha}} C^{\tilde{J}}{}_{Q\hat{\beta}} \mathcal{M}^{QM}\mathcal{M}^{PN} J_M{}^{\hat{\alpha}} J_N{}^{\hat{\beta}}$$

Uses the representation with index $\tilde{I}$ furnished by ancillary gauge transformation. Generalises extra E_8 term

E_{11} ExFT pseudo-Lagrangian

all fields

$$\mathcal{L}_{\mathsf{kin}} = \frac{1}{4}\mathcal{M}_{IJ}C^{IM}{}_{\widehat{\alpha}}C^{JN}{}_{\widehat{\beta}}J_M{}^{\widehat{\alpha}}J_N{}^{\widehat{\beta}} - \frac{1}{2}\mathcal{M}_{IJ}C^{IM}{}_{\widehat{\alpha}}C^{JN}{}_{\widehat{\Lambda}}J_M{}^{\widehat{\alpha}}\zeta_N{}^{\widehat{\Lambda}} - \frac{1}{4}\mathcal{M}_{IJ}C^{IM}{}_{\widehat{\Lambda}}C^{JN}{}_{\widehat{\Xi}}\zeta_M{}^{\widehat{\Lambda}}\zeta_N{}^{\widehat{\Xi}} = \frac{1}{4}\mathcal{M}_{IJ}F^IF^J + O(\zeta)$$

E_{11} ExFT pseudo-Lagrangian

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For topological term (no explicit $\mathcal{M}$ dependence) take inspiration from E_9 ExFT [Bossard, Ciceri Inverso, AK, Samtleben]

$$\mathcal{L}_{\text{top}} = \frac{1}{2}\Pi_{\tilde{\alpha}}{}^{MN}\Big(2\partial_{[M}\chi_{N]}{}^{\tilde{\alpha}} + J_{[M}{}^{\alpha}T_{\alpha}{}^{\tilde{\alpha}}{}_{\tilde{\beta}}\chi_{N]}{}^{\tilde{\beta}} + J_M{}^{\alpha}K_{[\alpha}{}^{\tilde{\alpha}}{}_{\beta]}J_N{}^{\beta}\Big)$$

$$- \frac{1}{2}\Omega_{IJ}C^{IM}{}_{\widehat{\alpha}}C^{JN}{}_{\widehat{\Lambda}}J_M{}^{\widehat{\alpha}}\zeta_N{}^{\widehat{\Lambda}}$$

all fields

First line is rigid E_{11}-invariant $d\chi$ total derivative

E_{11} ExFT

Pseudo-Lagrangian $\mathcal{L}_{E_{11}}$

- is gauge-invariant: $\delta_\xi \mathcal{L}_{E_{11}} = \partial_M \left(\xi^M \mathcal{L}_{E_{11}}\right)$
- combination of terms fixed by this requirement. Split somewhat artificial
- when varied w.r.t. constrained fields produces subset of duality equation $\mathcal{M}_{IJ} F^J = \Omega_{IJ} F^J$ $\Rightarrow$ consistent ✓
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Question: How does this describe $D = 11$ supergravity?

E_{11} ExFT and $D = 11$ SUGRA

Write pseudo-Lagrangian on $GL(11)$ solution to section constraint

$$\mathcal{L}_{E_{11}} = \sqrt{-g}\Big(R - \frac{1}{2\cdot 4!}\mathcal{F}_{n_1\ldots n_4}\mathcal{F}^{n_1\ldots n_4}\Big) - \frac{1}{144^2}\varepsilon^{n_1\ldots n_{11}}A_{n_1n_2n_3}\mathcal{F}_{n_4\ldots n_7}\mathcal{F}_{n_8\ldots n_{11}}$$

$$+\,\partial(\cdots) + \sum_{k=2}^{\infty}\big|\mathcal{E}_{I_{(k)}}\big|^2 \longleftarrow$$ can be ignored with duality equation

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Produces exactly $D = 11$ SUGRA equations of motion

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Note: This does not show E_{11} invariance of $D = 11$ SUGRA. Broken by solution to section constraint

Conclusions

- Constructed pseudo-Lagrangian and duality equations invariant under E_{11} generalised diffeomorphisms
- Ingredients: section constraint, extra constrained fields
- Reduces to all known SUGRAS/ExFTs
- Dual gravity realised sim. to [West] [Boulanger Hohm]
- Some remaining assumptions about E_{11} representations (η_{IJ}, ‘master’ identity)

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Thank you for your attention