$\mathcal{N}=2$ and $\mathcal{N}=4$ hyperbolic Calogero-Sutherland systems from gauging matrix models

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based on

SF, E. Ivanov, O. Lechtenfeld, Nucl. Phys. B944 (2019) 114633;
SF, Nucl. Phys. B953 (2020) 114977; B961 (2020) 115234

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Motivation

Multi-particle Calogero-Moser-Sutherland systems [F. Calogero, 1969, 1971; J. Moser, 1970; B. Sutherland, 1971, 1972] occupy a distinguished place among the integrable systems.

Supersymmetric generalizations of the Calogero-Moser-Sutherland models are of particular interest among possible developments.

Most of the researches in these directions have been devoted to supersymmetrization of the rational Calogero systems (see, for example, [D.Freedman, P.Mende, 1990;
L.Brink, T.Hansson, S.Konstein, M.Vasiliev, 1993;
S.Bellucci, A.Galajinsky, S.Krivonos, 2003; S.Bellucci, A.Galajinsky, E.Latini, 2005;
N.Wyllard, 2000; A.Galajinsky, O.Lechtenfeld, K.Polovnikov, 2006, 2008;
SF, E.Ivanov, O.Lechtenfeld, 2009, 2020; S.Krivonos, O.Lechtenfeld, 2011, 2020;
SF, E.Ivanov, 2016; SF, E.Ivanov, O.Lechtenfeld, S.Sidorov, 2018;
S.Krivonos, O.Lechtenfeld, A.Sutulin, 2018, 2019, 2020;
S.Krivonos, O.Lechtenfeld, A.Provorov, A.Sutulin, 2018; G.Antoniou, M.Feigin, 2019]
and the review [SF, E.Ivanov, O.Lechtenfeld, 2012]).

Supersymmetric generalizations of the hyperbolic and trigonometric Calogero-Sutherland systems have been studied in a very limited number of works (see, for example,
[B.Sriram Shastry, B.Sutherland, 1993; L.Brink, A.Turbiner, N.Wyllard, 1998;
A.Bordner, N.Manton, R.Sasaki, 2000; M.Ioffe, A.Neelov, 2000;
P.Desrosiers, L.Lapointe, P.Mathieu, 2001; A.Sergeev, 2002,
A.Sergeev, A.Veselov, 2004, 2017; S.Krivonos, O.Lechtenfeld, A.Sutulin, 2018, 2019, 2020;
S.Krivonos, O.Lechtenfeld, A.Provorov, A.Sutulin, 2018; G.Antoniou, M.Feigin, 2019;
S.Krivonos, O.Lechtenfeld, A.Sutulin, 2019, 2020]).

Supersymmetrization of the Calogero-Sutherland model implies the expansion of its phase space due to an additional set of odd (fermionic) variables.

In the standard scheme, the $\mathcal{N}$-supersymmetric extension of the n-particle Calogero-Sutherland system is achieved by introducing $\mathcal{N}n$ additional fermions.

However, beyond $\mathcal{N}=2$, serious difficulties arise in such extensions of n-particle models (see, e.g., [N.Wyllard, 2000; A.Galajinsky, O.Lechtenfeld, K.Polovnikov, 2006, 2008; G.Antoniou, M.Feigin, 2019]).

A different approach was put forward in [SF, E.Ivanov, O.Lechtenfeld, 2009], where $\mathcal{N}=1,2,4$ supersymmetric extensions of the rational Calogero model were derived by a gauging procedure [F.Delduc, E.Ivanov, 2006, 2007] applied to matrix superfield systems. Gauging formulation of pure bosonic Calogero-like model was considered in [A.Polychronakos, 1991; A.Gorsky, N.Nekrasov, 1994].

Such a gauging superfield approach was applied for obtaining new superconformal Calogero-Moser systems with deformed supersymmetry and intrinsic mass parameter [SF, E.Ivanov, O.Lechtenfeld, S.Sidorov, 2018].

A characteristic feature of the gauging approach is the presence of extra fermionic fields (as compared to a minimal extension) and, in the $\mathcal{N}=4$ case, of bosonic semi-dynamical spinning variables.

Within the Hamiltonian formalism, such type of matrix systems with an extended sets of fermionic fields was further used in [S.Krivonos, O.Lechtenfeld, A.Sutulin, 2018, 2019, 2020] for deriving supersymmetric Calogero systems.

In this talk we will consider the using of the gauging procedure in superfield matrix systems to obtain supersymmetric extensions of the hyperbolic A_{n-1} Calogero-Sutherland model, up to the $\mathcal{N}=4$ case.

Advantages of the gauging approach used:

- Starting from superfield model guarantees the presence of supersymmetry at all stages (if the performed procedures are supersymmetric).
- Since the bosonic limit describes the standard hyperbolic Calogero-Sutherland system, the superfield system is automatically a supersymmetrization of the required system.
- At all steps we have physically equivalent systems.
- Generators of nonlinear (complex) supersymmetry transformations are obtained from generators of standard linear transformations as a result of gauge fixings.

New properties of the gauging models:

- The models have semi-dynamical degrees of freedom. In the $\mathcal{N}=2$ case, they are pure gauge ones. But in the $\mathcal{N}=4$ case, some of them are dynamical and describe the spinning degrees of freedom.
- Since our initial models are superfield $(n\times n)$-matrix models, the number of fermionic degrees of freedom is proportional to n^2, in contrast to the more standard models with $\sim n$ fermions.

  It is possible that such an extension of the odd sector is necessary for obtaining supersymmetric generalization in the $\mathcal{N}\geq 4$ cases.

Hyperbolic Calogero-Sutherland model by gauging

Let us consider the formulation of the n-particle hyperbolic Calogero-Sutherland model as the $\mathrm{U}(n)$-gauging Hermitian matrix system.

The matrix model we will deal with is underlaid by the positive definite Hermitian $n \times n$–matrix field

$$X(t) := \|X_a{}^b(t)\|\,, \qquad (X_a{}^b)^* = X_b{}^a\,, \qquad \det X \neq 0\,, \qquad a,b = 1,\ldots,n\,,$$

and the complex $\mathrm{U}(n)$-spinor field

$$Z(t) := \|Z_a(t)\|\,, \qquad \bar{Z}^a = (Z_a)^*\,.$$

It also involves n^2 gauge fields

$$A(t) := \|A_a{}^b(t)\|\,, \qquad (A_a{}^b)^* = A_b{}^a\,.$$

The gauge-invariant action has the following form

$$\boxed{S = \int \mathrm{d}t \left[\frac{1}{2}\,\mathrm{Tr}\Big(X^{-1}\nabla X\, X^{-1}\nabla X\Big) + \frac{i}{2}\left(\bar{Z}\nabla Z - \nabla\bar{Z} Z\right) + c\,\mathrm{Tr}A\right],}$$

where the covariant derivatives are

$$\nabla X = \dot{X} + i\,[A,X]\,, \qquad \nabla Z = \dot{Z} + iAZ\,, \qquad \nabla\bar{Z} = \dot{\bar{Z}} - i\bar{Z}A\,.$$

The last Fayet-Iliopoulos term includes only $U(1)$ gauge field, c being a real constant.

The action is invariant with respect to the local $U(n)$ transformations, $g(\tau) \in U(n)$,

$$X \to gXg^{\dagger}\,, \qquad Z \to gZ\,, \quad \bar{Z} \to \bar{Z}g^{\dagger}\,, \qquad A \to gAg^{\dagger} + i\dot{g}g^{\dagger}\,.$$

Using these gauge transformations we can impose the gauge-fixing

$$X_a{}^b = 0\,, \quad a \neq b \qquad \Rightarrow \qquad X_a{}^b = x_a \delta_a{}^b\,,$$
$$\bar{Z}^a = Z_a\,.$$

As a result, in this gauge the final gauge-fixed action becomes

$$S = \frac{1}{2}\int \mathrm{d}t \left[\sum_a \frac{\dot{x}_a\dot{x}_a}{(x_a)^2} - \sum_{a\neq b}\frac{x_a x_b\, c^2}{(x_a - x_b)^2}\right].$$

Introducing the variables q_a as

$$x_a = \exp(q_a)\,,$$

we cast the action in the form

$$\boxed{S = \frac{1}{2}\int \mathrm{d}t \left[\sum_a \dot{q}_a\dot{q}_a - \sum_{a\neq b}\frac{c^2}{4\sinh^2\frac{q_a-q_b}{2}}\right]},$$

which is just the standard action of the hyperbolic Calogero-Sutherland system of the A_{n-1}-root type.

$\mathcal{N}=2$ and $\mathcal{N}=4$ superfield models

$\mathcal{N}=2$	$\mathcal{N}=4$
$\mathcal{N}=2$ s-space: $(t,\theta,\bar\theta)$, $\bar\theta=(\theta)^*$	$\mathcal{N}=4$ s-space: $(t,\theta^i,\bar\theta_i)$, $\bar\theta_i=(\theta^i)^*$, $i=1,2$
$\mathcal{N}=2$ chiral s-spaces: (t_L,θ), $(t_R,\bar\theta)$, $t_{L,R}=t\pm i\theta\bar\theta$	$\mathcal{N}=4$ harmonic s-space: $(t,\theta^\pm,\bar\theta^\pm,u_i^\pm)$ $\theta^\pm=\theta^i u_i^\pm\,,\bar\theta^\pm=\bar\theta^i u_i^\pm\,,u^{+i}u_i^-=1$
	Harmonic analytic s-space: $(\zeta,u)=(t_A,\theta^+,\bar\theta^+,u_i^\pm)$ $t_A=t+i(\theta^+\bar\theta^-+\theta^-\bar\theta^+)$
$\mathcal{N}=2$ invariant integration measure: $\mu=\mathrm{d}t\mathrm{d}^2\theta$	$\mathcal{N}=4$ invariant integration measures: $\mu_H=\mathrm{d}u\mathrm{d}t\mathrm{d}^4\theta,\quad \mu_A^{(-2)}=dudt_A\mathrm{d}\theta^+\mathrm{d}\bar\theta^+$
$\mathcal{N}=2$ covariant derivatives: $D=\partial_\theta-i\bar\theta\partial_t\,,\quad \bar D=-\partial_{\bar\theta}+i\theta\partial_t$	$\mathcal{N}=4$ covariant derivatives: $D^+=\frac{\partial}{\partial\theta^-}\,,\quad D^-=-\frac{\partial}{\partial\theta^+}-2i\bar\theta^-\partial_{t_A}$ $\bar D^+=-\frac{\partial}{\partial\bar\theta^-}\,,\quad \bar D^-=\frac{\partial}{\partial\bar\theta^+}-2i\theta^-\partial_{t_A}$ $D^{\pm\pm}=\partial^{\pm\pm}+2i\theta^\pm\bar\theta^\pm\partial_{t_A}+\theta^\pm\frac{\partial}{\partial\theta^\mp}+\bar\theta^\pm\frac{\partial}{\partial\bar\theta^\mp}$

Superfield contents

We gauge the action of n^2 superfields forming Hermitian $n \times n$ matrix $\mathcal{X} = \|\mathcal{X}_a{}^b\|$, $a = 1, \ldots, n$, and describing n multiplets $(1, \mathcal{N}, \mathcal{N}-1)$.

We use also n superfields forming SU(n) spinor $\Phi = \|\Phi_a\|$ and describing n semi-dynamical multiplets $(\mathcal{N}, \mathcal{N}, 0)$.

$\mathcal{N}=2$	$\mathcal{N}=4$
$\mathcal{N}=2$ prepotential $V_a{}^b(t,\theta,\bar\theta)$, $V = V^\dagger$ defines covariant derivatives $\mathcal{D}\mathcal{X} = D\mathcal{X} + \mathrm{e}^{-2V}(D\mathrm{e}^{2V})\,\mathcal{X}$, $\bar{\mathcal{D}}\mathcal{X} = \bar{D}\mathcal{X} - \mathcal{X}\,\mathrm{e}^{2V}(\bar{D}\mathrm{e}^{-2V})$	$\mathcal{N}=4$ connection $V^{++}{}_a{}^b(\zeta,u)$, $V^{++} = \tilde{V}^{++}$ defines harmonic covariant derivatives $\mathcal{D}^{++} = D^{++} + i\,V^{++}$ and all other connections
Hermitian $n \times n$ matrix $\mathcal{N}=2$ superfield $\mathcal{X}_a{}^b(t,\theta,\bar\theta)$, $(\mathcal{X})^\dagger = \mathcal{X}$	Hermitian $n \times n$ matrix $\mathcal{N}=4$ superfield $\mathcal{X}_a{}^b(t,\theta^\pm,\bar\theta^\pm,u)$, $\tilde{\mathcal{X}} = \mathcal{X}$ subject to the constraints $\mathcal{D}^{++}\,\mathcal{X} = 0$, $\mathcal{D}^+\mathcal{D}^-\mathcal{X} = \bar{\mathcal{D}}^+\bar{\mathcal{D}}^-\mathcal{X} = (\mathcal{D}^+\bar{\mathcal{D}}^- + \bar{\mathcal{D}}^+\mathcal{D}^-)\mathcal{X} = 0$
even U(n)-spinor superfield $\mathcal{Z}_a(t_L,\theta)$ and its conjugate $\bar{\mathcal{Z}}^a(t_R,\bar\theta)$ are commuting (anti)chiral $\mathcal{N}=2$ superfields satisfying $\bar{D}\mathcal{Z}_a = 0$, $D\bar{\mathcal{Z}}^a = 0$	even U(n)-spinor superfield $\mathcal{Z}^+_a(\zeta,u)$ and its conjugate $\tilde{\mathcal{Z}}^{+a}(\zeta,u)$ are commuting analytic $\mathcal{N}=4$ superfields satisfying the constraints $\mathcal{D}^{++}\,\mathcal{Z}^+ = 0$, $\mathcal{D}^+\,\mathcal{Z}^+ = 0$, $\bar{\mathcal{D}}^+\,\mathcal{Z}^+ = 0$

Superfield actions

$\mathcal{N}$-supersymmetric and U(n) gauge invariant action

$$S = S_{\mathfrak{X}} + S_{WZ} + S_{FI}\,.$$

is the sum of the action $S_{\mathfrak{X}}$ for dynamical supermuptiplet $\mathfrak{X}$, Wess-Zumuno term S_{WZ} for semi-dynamical supermultiplet $\mathcal{Z}$ and Fayet-Iliopoulos term S_{FI} for gauge supermultiplet V.

$\mathcal{N}=2$	$\mathcal{N}=4$
$S_{\mathfrak{X}} = \frac{1}{2}\int \mu\,\mathrm{Tr}\left(\mathfrak{X}^{-1}\bar{\mathcal{D}}\mathfrak{X}\,\mathfrak{X}^{-1}\mathcal{D}\mathfrak{X}\right)$	$S_{\mathfrak{X}} = \frac{1}{2}\int \mu_H\,\mathrm{Tr}\left(\ln\mathfrak{X}\right)$
$S_{WZ} = -\frac{1}{2}\int \mu\,\bar{\mathcal{Z}}\,\mathrm{e}^{2V}\mathcal{Z}$	$S_{WZ} = -\frac{1}{2}\int \mu_A^{(-2)}\,\widetilde{\mathcal{Z}}^{+}\,\mathcal{Z}^{+}$
$S_{FI} = c\int \mu\,\mathrm{Tr}V$	$S_{FI} = -\frac{ic}{2}\int \mu_A^{(-2)}\,\mathrm{Tr}\left(V^{++}\right)$

c is a real constant.

The action is invariant with respect to the local $\mathrm{U}(n)$ transformations:

$\mathcal{N}=2$	$\mathcal{N}=4$
$\mathcal{X}' = e^{i\lambda}\,\mathcal{X}\,e^{-i\bar\lambda}$, $\quad \mathcal{Z}' = e^{i\lambda}\mathcal{Z}, \quad \bar{\mathcal{Z}}' = \bar{\mathcal{Z}}\,e^{-i\bar\lambda}$, $\quad e^{2V'} = e^{i\bar\lambda}\,e^{2V}e^{-i\lambda}$, $\quad \lambda(t_L,\theta)\in u(n), \quad \bar\lambda(t_R,\theta) = (\lambda)^\dagger \in u(n)$	$\mathcal{X}' = e^{i\lambda}\,\mathcal{X}\,e^{-i\lambda}$, $\quad \mathcal{Z}^{+\prime} = e^{i\lambda}\mathcal{Z}^+, \quad \widetilde{\mathcal{Z}}^{+\prime} = \widetilde{\mathcal{Z}}^+\,e^{-i\lambda}$, $\quad V^{++\,\prime} = e^{i\lambda}\,V^{++}\,e^{-i\lambda} + i\,e^{i\lambda}(D^{++}e^{-i\lambda})$, $\quad \lambda(\zeta,u^\pm)\in u(n), \quad \widetilde{\lambda} = \lambda$

We can choose Wess-Zumino gauge

$\mathcal{N}=2$	$\mathcal{N}=4$
$V_a{}^b \;=\; \theta\bar\theta\, A_a{}^b(t)$	$V^{++}{}_a{}^b \;=\; 2i\,\theta^+\bar\theta^+\, A_a{}^b(t_A)$

Physical fields are presented in the expansions

$\mathcal{N}=2$	$\mathcal{N}=4$
$\mathcal{X}_a{}^b \;=\; X_a{}^b \;+\; \theta\Psi_a{}^b \;-\; \bar\theta\bar\Psi_a{}^b \;+\ldots$	$\mathcal{X}_a{}^b \;=\; X_a{}^b \;+\; \theta^-\Psi^i{}_a{}^b u_i^\pm \;+\; \bar\theta^-\bar\Psi^i{}_a{}^b u_i^\pm \;+\ldots$
where $\Psi_a{}^b$, $\bar\Psi_a{}^b = (\Psi_b{}^a)^*$ – odd fields	where $\Psi^i{}_a{}^b$, $\quad \bar\Psi^i{}_a{}^b = (\Psi_{ib}{}^a)^*$ – odd fields
$\mathcal{Z}_a \;=\; Z_a \;+\ldots, \quad \bar{\mathcal{Z}}^a \;=\; \bar Z^a \;+\ldots$	$\mathcal{Z}^+_a = Z^i_a u^+_i + \ldots, \quad \widetilde{\mathcal{Z}}^{+a} = \bar Z^a_i u^{+i} + \ldots, \quad \bar Z^a_i = \overline{(Z^i_a)}$

After putting the Wess-Zumino gauge and elimination of auxiliary fields, we obtain the on-shell component action $S_{\rm matrix} = \int dt\, L_{\rm matrix}$ with the Lagrangian

$$L_{\rm matrix} = L_{\rm b} + L_{\rm 2f} + L_{\rm 4f} \,,$$

$$\begin{aligned} L_{\rm b} &= \frac{1}{2}\,{\rm Tr}\Big(X^{-1}\nabla X\, X^{-1}\nabla X + 2c\,A\Big) + \frac{i}{2}\left(\bar Z_k \nabla Z^k - \nabla\bar Z_k Z^k\right) \\ L_{\rm 2f} &= \frac{i}{2}\,{\rm Tr}\Big(X^{-1}\bar\Psi_k X^{-1}\nabla\Psi^k - X^{-1}\nabla\bar\Psi_k X^{-1}\Psi^k\Big) \\ L_{\rm 4f} &= \frac{1}{8}\,{\rm Tr}\Big(\{X^{-1}\Psi^i, X^{-1}\bar\Psi_i\}\,\{X^{-1}\Psi^k, X^{-1}\bar\Psi_k\}\Big)\,, \end{aligned}$$

where the quantity c is a real constant and the covariant derivatives are defined by $\nabla X = \dot X + i\,[A,X]$ and $\nabla\Psi^k = \dot\Psi^k + i\,[A,\Psi^k]$, $\nabla Z^k = \dot Z^k + iAZ^k$ and c.c.

Properties:

$\mathcal{N}=2$	$\mathcal{N}=4$
index k takes only one value $k=1$	index k is $SU(2)$-index and runs over two values $k=1,2$
L_b is exactly the Lagrangian of the hyperbolic Calogero-Sutherland system	L_b describes the $SU(2)$ spin hyperbolic Calogero-Sutherland system (will see below)
$\Downarrow$	$\Downarrow$
$S_{\rm matrix}$ describes $\mathcal{N}=2$ supersymmetric hyperbolic Calogero-Sutherland system	$S_{\rm matrix}$ describes $\mathcal{N}=4$ supersym. $SU(2)$ spin hyperbolic Calogero-Sutherland system

Hamiltonian formulation

The matrix system is described by total Hamiltonian

$$H_{\rm matrix} \; = \; H + \mathrm{Tr}\big(AF\big)\,,$$

where the first term is the canonical Hamiltonian

$$H \; = \; \frac{1}{2}\,\mathrm{Tr}\Big(XPXP\Big) \; - \; \frac{1}{8}\,\mathrm{Tr}\Big(\{X^{-1}\Psi^i, X^{-1}\bar\Psi_i\}\,\{X^{-1}\Psi^k, X^{-1}\bar\Psi_k\}\Big)$$

and the second term $\mathrm{Tr}\big(AF\big)$ uses the quantities

$$F_a{}^b := i[P,X]_a{}^b + Z^k_a \bar Z^b_k - \frac{1}{2}\,\{X^{-1}\Psi^k, X^{-1}\bar\Psi_k\}_a{}^b - \frac{1}{2}\,\{\Psi^k X^{-1}, \bar\Psi_k X^{-1}\}_a{}^b - c\,\delta_a{}^b\,.$$

The variables $A_a{}^b$ play the role of the Lagrange multipliers for the constraints

$$F_a{}^b \approx 0\,.$$

Due to the 2-nd class constraints also present here, the nonvanishing Dirac brackets take the form

$$\{X_a{}^b, P_c{}^d\}_{\rm D} = \delta^d_a \delta^b_c\,, \qquad \{Z^i_a, \bar Z^b_k\}_{\rm D} = -i\,\delta^b_a \delta^i_k\,, \qquad \{\Psi^i{}_a{}^b, \bar\Psi_{k\,c}{}^d\}_{\rm D} = -iX_a{}^d X_c{}^b \delta^i_k\,.$$

But there are also the nonvanishing Dirac brackets

$$\{P_a{}^b, P_c{}^d\}_{\rm D} \;\sim\; (X^{-1}\Psi^k X^{-1}\bar\Psi_k)_a{}^d X^{-1b}_{\;c} \;\neq 0\,,$$

$$\{\Psi^k{}_a{}^b, P_c{}^d\}_{\rm D} \;\sim\; \delta^d_a (X^{-1}\Psi^k)_c{}^b \;\neq 0\,, \qquad \{\bar\Psi_{k\,a}{}^b, P_c{}^d\}_{\rm D} \;\sim\; \delta^d_a (X^{-1}\bar\Psi_k)_c{}^b \;\neq 0\,.$$

Partial gauge-fixing of the matrix system

The constraints $F_a{}^b = (F_b{}^a)^*$ form the $u(n)$ algebra with respect to the Dirac brackets,

$$\{F_a{}^b, F_c{}^d\}_{\rm D} = -i\delta_a{}^d F_c{}^b + i\delta_c{}^b F_a{}^d \,,$$

are the 1-st class ones and generate local U(n) transformations

$$\delta X_a{}^b = -i[\alpha, X]_a{}^b \,, \quad \delta P_a{}^b = -i[\alpha, P]_a{}^b \,, \quad \delta Z_a^k = -i(\alpha Z^k)_a \,, \delta \Psi^k{}_a{}^b = -i[\alpha, \Psi^k]_a{}^b \,,$$

where $\alpha_a{}^b(\tau) = (\alpha_b{}^a(\tau))^*$ are the local parameters.

Below we apply the expansions $X_a{}^b = x_a \delta_a{}^b + x_a{}^b$, $\qquad P_a{}^b = \mathrm{p}_a \delta_a{}^b + \mathrm{p}_a{}^b$, where $x_a{}^b$ and $\mathrm{p}_a{}^b$ represent the off-diagonal matrix quantities.

The gauges $X_a{}^b = 0$ at $a \neq b$ fix the local transformations with the parameters $\alpha_a{}^b(\tau)$, $a \neq b$ generated by the off-diagonal constraints $F_a{}^b \approx 0$, $a \neq b$. This gauge fixing takes the form

$$x_a{}^b \approx 0$$

In addition, using the constraints $F_a{}^b \approx 0$, $a \neq b$, we express the momenta $\mathrm{p}_a{}^b$ through the remaining phase variables:

$$\mathrm{p}_a{}^b = -\frac{i\, Z_a^k \bar{Z}_k^b}{x_a - x_b} + \frac{i\,(x_a + x_b)\, \{\Phi^k, \bar{\Phi}_k\}_a{}^b}{2(x_a - x_b)\sqrt{x_a x_b}} \,,$$

where we use the odd matrix variables defined by

$$\Phi^k{}_a{}^b := \frac{\Psi^k{}_a{}^b}{\sqrt{x_a x_b}} \,, \qquad \bar{\Phi}_k{}_a{}^b = (\Phi^k{}_b{}^a)^* := \frac{\bar{\Psi}_k{}_a{}^b}{\sqrt{x_a x_b}} \,.$$

Final physical systems

For obtaining standard Hamiltonian $\sim \sum_a (p_a)^2$ we introduce the phase variables

$$q_a = \log x_a \,, \quad p_a = x_a \mathrm{p}_a \,.$$

As a result, after the partial gauge fixing, phase spaces of the considered systems are defined by

• $2n$ even real variables q_a, p_a, • $\mathcal{N}n^2$ odd complex variables $\Phi^i{}_a{}^b$,

• $\mathcal{N}n/2$ even complex variables Z^i_a, which nonvanishing Dirac brackets are

$$\boxed{\{q_a, p_b\}'_{\mathrm{D}} = \delta_{ab} \,, \qquad \{Z^i_a, \bar{Z}^b_k\}'_{\mathrm{D}} = -i\, \delta^b_a \delta^i_k \,, \qquad \{\Phi^i{}_a{}^b, \bar{\Phi}_k{}_c{}^d\}'_{\mathrm{D}} = -i\, \delta^d_a \delta^b_c \delta^i_k \,.}$$

In these variables the Hamiltonians take the form

$$\boxed{\mathrm{H} \;=\; \frac{1}{2} \sum_a p_a p_a \;+\; \frac{1}{8} \sum_{a \neq b} \frac{R_a{}^b R_b{}^a}{\sinh^2 \left(\frac{q_a - q_b}{2} \right)} \;-\; \frac{1}{8}\, \mathrm{Tr}\Big(\{\Phi^i, \bar{\Phi}_i\} \{\Phi^k, \bar{\Phi}_k\} \Big) \,,}$$

$$\text{where} \qquad R_a{}^b := Z^k_a \bar{Z}^b_k - \cosh\left(\frac{q_a - q_b}{2} \right) \{\Phi^k, \bar{\Phi}_k\}_a{}^b \,.$$

The residual 1-st class constraints are n diagonal constraints

$$\boxed{F_a := F_a{}^a = R_a{}^a - c = Z^k_a \bar{Z}^a_k - \{\Phi^k, \bar{\Phi}_k\}_a{}^a - c \approx 0 \qquad (\text{no summation over } a) \,,}$$

which form an abelian algebra with respect to the Dirac brackets $\{F_a, F_b\}'_{\mathrm{D}} = 0$ and generate the $[\mathrm{U}(1)]^n$ gauge transformations with the local parameters $\gamma_a(t)$:

$$Z^k_a \to \mathrm{e}^{i\gamma_a} Z^k_a \,, \qquad \Phi^k{}_a{}^b \to \mathrm{e}^{i\gamma_a} \Phi^k{}_a{}^b \mathrm{e}^{-i\gamma_b} \qquad (\text{no sums over } a, b) \,.$$

Properties of the $\mathcal{N}$-supersymmetric systems obtained

- Since the starting systems are $\mathcal{N}$-supersymmetric matrix models, the resulting systems are described by $\mathcal{N}n^2$ fermions, in contrast to the standard supersymmetric Calogero-like system [D.Freedman, P.Mende, 1990] which involves only $\mathcal{N}n$ fermions.

- $\mathcal{N}$-supersymmetric systems involve $\mathcal{N}n/2$ semi-dynamical even variables Z.
  In $\mathcal{N}{=}2$ case, there are n 1-st class constraints which allow us to gauging all these variables Z_a.

- $\mathcal{N}{=}4$ system uses SU(2) spinors Z_a^i and in bosonic limit its Hamiltonian has the form

$$\mathrm{H}_{bose} \;=\; \frac{1}{2}\sum_a p_a p_a + \frac{1}{8}\sum_{a\neq b}\frac{\mathrm{Tr}(S_a S_b)}{\sinh^2\left(\dfrac{q_a - q_b}{2}\right)}\,,$$

where the $n{\times}n$ matrix quantities $\quad {S_a}_i{}^k := \bar{Z}_i^a Z_a^k \quad$ at all values a form the $u(2)$ algebras with respect to the Dirac brackets:

$$\{{S_a}_i{}^k, {S_b}_j{}^l\}'_{\mathrm{D}} = -i\,\delta_{ab}\left(\delta_j^k {S_a}_i{}^l - \delta_i^l {S_a}_j{}^k\right)\,.$$

Thus, the Hamiltonian is same as the Hamiltonian of the U(2)-spin hyperbolic Calogero-Sutherland A_{n-1}-root system [J.Gibbons, T.Hermsen, 1984; S.Wojciechowski, 1985]. Hence we built $\mathcal{N}{=}4$ supersymmetric $SU(2)$ spin hyperbolic C-S system.

Supersymmetry generators

Since the systems considered here are obtained from $\mathcal{N}$-superfield action, they possess $\mathcal{N}$-supersymmetry invariance.

Putting the partial gauge fixing conditions in expressions of the Noether charges, we obtain supersymmetry generators

$$\boxed{\begin{aligned} \mathrm{Q}^i &= \sum_a p_a \Phi^i{}_a{}^a - \frac{i}{2}\sum_{a\neq b} \frac{R_a{}^b \Phi^i{}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)} + \frac{i}{2}\sum_{a,b} [\Phi^k,\bar\Phi_k]_a{}^b \Phi^i{}_b{}^a\,, \\ \bar{\mathrm{Q}}_i &= \sum_a p_a \bar\Phi_{ia}{}^a - \frac{i}{2}\sum_{a\neq b} \frac{R_a{}^b \bar\Phi_{ib}{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)} - \frac{i}{2}\sum_{a,b} [\Phi^k,\bar\Phi_k]_a{}^b \bar\Phi_{ib}{}^a \end{aligned}}$$

where $\qquad R_a{}^b := Z^k_a \bar Z^b_k - \cosh\left(\frac{q_a-q_b}{2}\right)\{\Phi^k,\bar\Phi_k\}_a{}^b\,.$

- In $\mathcal{N}=2$ case, index i takes only one value $i=1$ and last terms are identically zero.
- Second terms in the supercharges describing the Calogero-like interaction are zero when the off-diagonal matrix fermions $\Phi^i{}_a{}^b$, $\bar\Phi_{ia}{}^b$, $a\neq b$ vanish.

  $\Downarrow$

  The matrix nature of the original system is important for the given type of systems.

Supercharges Q^i, $\bar{\mathrm{Q}}_i$ and Hamiltonian H form the $\mathcal{N}=2$ or $\mathcal{N}=4$ superalgebra with respect to the Dirac brackets on the shell of the 1-st class constraints $F_a \approx 0$:

$$\{\mathrm{Q}^i, \mathrm{Q}^k\}'_{\mathrm{D}} = -\frac{i}{4}\sum_{a\neq b} \frac{\Phi^{(i}{}_a{}^b \Phi^{k)}{}_b{}^a}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} \left(F_a - F_b\right),$$

$$\{\mathrm{Q}^i, \bar{\mathrm{Q}}_k\}'_{\mathrm{D}} = -2i\,\mathrm{H}\,\delta^i_k - \frac{i}{4}\sum_{a\neq b} \frac{\Phi^i{}_a{}^b \bar{\Phi}_k{}_b{}^a}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} \left(F_a - F_b\right),$$

$$\{\mathrm{Q}^i, \mathrm{H}\}'_{\mathrm{D}} = -\frac{1}{8}\sum_{a\neq b} \frac{R_a{}^b \Phi^i{}_b{}^a}{\sinh^3\left(\frac{q_a-q_b}{2}\right)} \left(F_a - F_b\right).$$

These generators H, Q^i, $\bar{\mathrm{Q}}_i$ are gauge invariant: $\{\mathrm{Q}^i, F_a\}'_{\mathrm{D}} = \{\bar{\mathrm{Q}}_i, F_a\}'_{\mathrm{D}} = \{\mathrm{H}, F_a\}'_{\mathrm{D}} = 0$.

Remark:
As already noted, in the $\mathcal{N}=2$ case it is possible to gauge-eliminate all semi-dynamical variables Z_a. As a result of this procedure, we reproduce supercharges [S.Krivonos, O.Lechtenfeld, A.Provorov, A.Sutulin, 2018, 2020]
that either have irrational dependence on odd variables
or use odd quantities with nontrivial properties with respect to complex conjugation.

Lax representation

Classical dynamics of the system considered here is defined by the total Hamiltonian

$$\mathrm{H_T} \;=\; \mathrm{H} \;+\; \textstyle\sum_a \lambda_a F_a \,,$$

where $\lambda_a(t)$ are the Lagrange multipliers for the 1-st class constraints F_a.
A time derivative of an arbitrary phase variable $B(t)$ takes the form $\dot{B} \;=\; \{B, \mathrm{H_T}\}'_{\mathrm{D}}$.

This evolution can be written using the $n{\times}n$ matrix Lax pair

$$L_a{}^b \;=\; p_a\,\delta_a{}^b \;-\; i\left(1-\delta_a^b\right)\frac{R_a{}^b}{2\sinh\left(\dfrac{q_a-q_b}{2}\right)}\,,$$

$$M_a{}^b \;=\; \frac{1}{4}\,\{\Phi^k,\bar{\Phi}_k\}_a{}^a\delta_a{}^b \;+\; \frac{1}{4}\left(1-\delta_a^b\right)\left(\frac{\cosh\left(\dfrac{q_a-q_b}{2}\right)}{\sinh^2\left(\dfrac{q_a-q_b}{2}\right)}\,R_a{}^b+\{\Phi^k,\bar{\Phi}_k\}_a{}^b\right)+\lambda_a\,\delta_a{}^b.$$

Evolution of (q,p)-variables is represented by the matrix commutator

$$\dot{L}_a{}^b \;=\; -i[M,L]_a{}^b - i\left(1-\delta_a^b\right)\frac{L_a{}^b\,(F_a-F_b)}{4\sinh^2\left(\dfrac{q_a-q_b}{2}\right)}\,,$$

where $F_a \approx 0$ are the constraints.

The equations of motion of odd matrix variables $\Phi^{i\,b}_{a}$, $\bar\Phi_{i\,a}{}^{b}$ are also represented as commutators

$$\dot\Phi^{i\,b}_{a} = -i[M,\Phi^{i}]_{a}{}^{b}\,, \qquad \dot{\bar\Phi}_{i\,a}{}^{b} = -i[M,\bar\Phi_{i}]_{a}{}^{b}\,.$$

The Lax equations yield the conserved charges:

the trace of any polynomial function

$$\mathcal{F}(L,\Phi,\bar\Phi)$$

of the matrix variables $L_a{}^b$, $\Phi^{i\,b}_{a}$, $\bar\Phi_{i\,a}{}^{b}$
is conserved quantity on the shell of constraints the 1-st class constraints $F_a \approx 0$:

$$\mathcal{J} := \mathrm{Tr}(\mathcal{F})\,, \qquad \dot{\mathcal{J}} \approx 0\,.$$

In particular, supercharges and Hamiltonian have the same form

$$\mathrm{Q}^i = \mathrm{Tr}(\Phi^i L) + \frac{i}{2}\,\mathrm{Tr}\Big([\Phi^k,\bar\Phi_k]\Phi^i\Big)\,, \qquad \bar{\mathrm{Q}}_i = \mathrm{Tr}(\bar\Phi_i L^{\mathrm{k}-1}) - \frac{i}{2}\,\mathrm{Tr}\Big(\bar\Phi_i[\Phi^k,\bar\Phi_k]\,,$$

$$\mathrm{H} = \frac{1}{2}\,\mathrm{Tr}(L^2) - \frac{1}{8}\,\mathrm{Tr}\Big(\{\Phi^i,\bar\Phi_i\}\{\Phi^k,\bar\Phi_k\}\Big)\,.$$

The equations of motion of the commuting spinning variables Z^i_a, $\bar{Z}^a_i$ are represented as

$$\dot{Z}^i_a = -i\sum_b A_a{}^b Z^i_b\,, \quad \dot{\bar{Z}}{}^a_i = i\sum_b \bar{Z}^b_i A_b{}^a\,, \ \text{where} \ A_a{}^b \ = \ \left(1-\delta^b_a\right)\frac{R_a{}^b}{4\sinh^2\left(\frac{q_a-q_b}{2}\right)} + \lambda_a\,\delta_a{}^b\,.$$

We obtain that the U(2) charges $\quad S_k{}^i := \sum_a \bar{Z}^a_k Z^i_a \quad$ are conserved: $\quad \dot{S}_k{}^i = 0.$

Moreover, $Z^{j=2}_a = 0$, $\bar{Z}^a_{j=2} = 0$, at all a are $\mathcal{N}{=}4$ supersymmetry invariant and can be used as reduction conditions. After reduction our $\mathcal{N}{=}4$ system involves only half of initial semi-dynamical variables $z_a := Z^{j=1}_a$, $\bar{z}^a := \bar{Z}^a_{j=1}$, $\{z_a, \bar{z}^b\}'_{\mathrm{D}} = -i\delta^b_a$ and describes $\mathcal{N}{=}4$ spinless hyperbolic Calogero-Sutherland system:

$$\mathcal{Q}^i = \sum_a p_a \Phi^i{}_a{}^a \ - \ \frac{i}{2}\sum_{a\neq b}\frac{T_a{}^b\Phi^i{}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)} \ + \ \frac{i}{2}\sum_{a,b}[\Phi^k,\bar{\Phi}_k]_a{}^b\Phi^i{}_b{}^a\,, \quad \bar{\mathcal{Q}}_i\,,$$

$$\mathcal{H} = \frac{1}{2}\sum_a p_a p_a + \frac{1}{8}\sum_{a\neq b}\frac{T_a{}^b T_b{}^a}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} - \frac{1}{8}\,\mathrm{Tr}\Big(\{\Phi^i,\bar{\Phi}_i\}\{\Phi^k,\bar{\Phi}_k\}\Big)\,,$$

where $\quad T_a{}^b := z_a\bar{z}^b - \cosh\left(\frac{q_a-q_b}{2}\right)\{\Phi^k,\bar{\Phi}_k\}_a{}^b \quad$ and with the 1-st class constraints

$$\mathcal{F}_a := T_a{}^a - c = z_a\bar{z}^a - \{\Phi^k,\bar{\Phi}_k\}_a{}^a - c \approx 0 \qquad \text{(no summation over } a\text{)}\,.$$

Gauge-elimination of all semi-dynamical variables z_a in the last system yields the $\mathcal{N}{=}4$ supercharges presented in [S.Krivonos, O.Lechtenfeld, 2020] with nontrivial properties of odd variables with respect to complex conjugation.

Quantum generators

Supersymmetry quantum generators are obtained by using the Weyl ordering of supersymmetry classical ones. Let us present the $\mathcal{N}=2$ case.

Quantum $\mathcal{N}=2$ supersymmetric hyperbolic Calogero-Sutherland system is described by the operators $\mathbf{q}_a$, $\mathbf{p}_a$, $\mathbf{Z}_a$, $\bar{\mathbf{Z}}^a$, $\mathbf{\Phi}_a{}^b$, $\bar{\mathbf{\Phi}}_a{}^b$, with canonical (anti)commutation relations:

$$[\mathbf{q}_a, \mathbf{p}_b] = i\,\delta_{ab}\,, \qquad [\mathbf{Z}_a, \bar{\mathbf{Z}}^b] = \delta_a^b\,, \qquad \{\mathbf{\Phi}_a{}^b, \bar{\mathbf{\Phi}}_c{}^d\} = \delta_a^d \delta_c^b\,.$$

Below we use the odd operators φ_a, $\phi_a{}^b$ and c.c. which are defined by the expansions

$$\mathbf{\Phi}_a{}^b = \varphi_a \delta_a{}^b + \phi_a{}^b\,, \quad \bar{\mathbf{\Phi}}_a{}^b = \bar\varphi_a \delta_a{}^b + \bar\phi_a{}^b\,, \quad \{\varphi_a, \bar\varphi_b\} = \delta_{ab}\,, \qquad \{\phi_a{}^b, \bar\phi_c{}^d\} = \delta_a^d \delta_c^b\,.$$

Quantum supercharges have the form

$$\begin{aligned}
\mathbf{Q} &= \sum_a \mathbf{p}_a \varphi_a - \frac{i}{4}\sum_{a\neq b} \coth\left(\frac{q_a-q_b}{2}\right)\left(\varphi_a - \varphi_b\right) - \frac{i}{2}\sum_{a\neq b} \frac{\mathbf{R}_a{}^b \phi_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)}\,, \\
\bar{\mathbf{Q}} &= \sum_a \mathbf{p}_a \bar\varphi_a + \frac{i}{4}\sum_{a\neq b} \coth\left(\frac{q_a-q_b}{2}\right)\left(\bar\varphi_a - \bar\varphi_b\right) + \frac{i}{2}\sum_{a\neq b} \frac{\bar\phi_a{}^b \mathbf{R}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)}\,, \\
\mathbf{H} &= \frac{1}{2}\sum_a \mathbf{p}_a \mathbf{p}_a + \frac{1}{8}\sum_{a\neq b} \frac{\mathbf{R}_a{}^b \mathbf{R}_b{}^a}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} - \frac{1}{8}\,\mathrm{Tr}\Big(\{\mathbf{\Phi},\bar{\mathbf{\Phi}}\}\{\mathbf{\Phi},\bar{\mathbf{\Phi}}\}\Big) + \frac{n(4n^2-1)}{24}\,.
\end{aligned}$$

$$\text{where} \qquad \mathbf{R}_a{}^b := \mathbf{Z}_a \bar{\mathbf{Z}}^b - \cosh\left(\frac{q_a-q_b}{2}\right)\{\mathbf{\Phi},\bar{\mathbf{\Phi}}\}_a{}^b$$

Physical states $|\Psi\rangle$ obey the conditions $\mathbf{F}_a|\Psi\rangle = 0$ where the operators

$$\mathbf{F}_a := \mathbf{R}_a{}^a - c = \mathbf{Z}_a\bar{\mathbf{Z}}^a - \{\mathbf{\Phi}, \bar{\mathbf{\Phi}}\}_a{}^a - c \qquad \text{(no summation over } a\text{)}$$

are quantum counterparts of the 1-st class constraints.

In the space of physical states $|\Psi\rangle$ the operators $\mathbf{Q}$, $\bar{\mathbf{Q}}$, $\mathbf{H}$ form the $\mathcal{N}{=}2$ superalgebra.

In contrast to the classical supercharges, the quantum supercharges have the following special property: the first two terms in them

$$\mathbb{Q} = \sum_a \mathbf{p}_a \varphi_a - \frac{i}{4}\sum_{a\neq b} \coth\left(\frac{q_a - q_b}{2}\right)\left(\varphi_a - \varphi_b\right), \qquad \text{and c.c.}$$

do not contain off-diagonal fermions $\phi_a{}^b$, $\bar{\phi}_a{}^b$ and themselves form the $\mathcal{N}{=}2$ superalgebra

$$\{\mathbb{Q}, \bar{\mathbb{Q}}\} = 2\,\mathbb{H}\,, \qquad \{\mathbb{Q}, \mathbb{Q}\} = \{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}\} = 0\,,$$

where the Hamiltonian of such a “truncated” subsystem is given by the following expression:

$$\mathbb{H} = \frac{1}{2}\sum_a \mathbf{p}_a\mathbf{p}_a + \frac{1}{8}\sum_{a\neq b} \frac{\left(\bar{\varphi}_a - \bar{\varphi}_b\right)\left(\varphi_a - \varphi_b\right)}{\sinh^2\left(\frac{q_a - q_b}{2}\right)} + \frac{n(n^2-1)}{24}\,.$$

Such system is in fact the $\mathcal{N}{=}2$ special extension of the hyperbolic Calogero-Sutherland system with a fixed value of the coupling constant.

In contrast to the $\mathcal{N}{=}2$ case, the separation of the invariant sector with only diagonal odd variables does not work in the $\mathcal{N}{=}4$ quantum case.

Conclusion

This talk presented new models of multi-particle supersymmetric mechanics, which are $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric generalizations of the Calogero-Sutherland hyperbolic system of A_{n-1} root type.

- Important element of our construction was superfield matrix systems with $\mathrm{U}(n)$ gauge symmetry.
- Resulting n-particle systems possess $\mathcal{N}n^2$ real physical fermions.
- In the $\mathcal{N}=4$ case, the system involves additional semi-dynamical bosonic spin variables and so describes a $\mathcal{N}=4$ supersymmetric generalization of the $\mathrm{U}(2)$-spin Calogero-Sutherland hyperbolic system.
- It is possible to impose the reduction conditions that are $\mathcal{N}=4$ supersymmetry invariant and eliminate half of the spinning variables. Such a reduced system is in fact the $\mathcal{N}=4$ generalization of the spinless hyperbolic Calogero-Sutherland system.
- Explicit expressions are obtained for the classical and quantum supersymmetry generators, corresponding to the hyperbolic Calogero-Sutherland system.
- The Lax representation of the equations of motion for the system under consideration is presented and the set of conserved quantities is found.

THANK YOU !