Integrable structures in CFT and affine Yangians

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- The program devoted to calculation of simultaneous spectra of IM's in CFT has been initiated by BLZ in 1994 (KdV system)

- AGT relation (2009) gave new insights (BO system)

- There is an interpolating integrable system called ILW. It's spectrum is found to be described by BAE (Nekrasov-Okounkov,A.L. 2013)

- BA has been proved by Feigin et all in $q-$deformed case (also by Aganagic and Okounkov)

- We suggested alternative proof in the conformal case

- generalized it to BCD CFT's

- generalized it to paraCFT ($N = 1$ Virasoro etc): $\mathsf{Y}(\widehat{\mathfrak{gl}(1)}) \to \mathsf{Y}(\widehat{\mathfrak{gl}(n)})$

Introduction

There is a large class of $2D$ QFT's defined by Toda action

$$S_0 = \int \left(\frac{1}{8\pi} \Big(\partial_\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi} \Big) + \Lambda \sum_{r=1}^{N} e^{\big(\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi} \big)} \right) \, d^2x.$$

This theory properly coupled to a background metric, defines a conformal field theory. However, it is well known, that under some conditions on the set $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$ it also enjoys enlarged conformal symmetry usually referred as $W-$algebra.

There is a class of such distinguishable sets $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$ with semi-classical behavior

$$\boldsymbol{\alpha}_r = b\boldsymbol{e}_r \quad \text{for all} \quad r = 1, \ldots, N,$$

where $\boldsymbol{e}_r$ are finite in the limit $b \to 0$. The vectors $\boldsymbol{e}_r$ have to be simple roots of a semi-simple Lie algebra $\mathfrak{g}$ of rank N.

An interesting question arises if one perturbs the theory by an additional exponential field

$$S_0 \to S_0 + \lambda \int e^{\left(\boldsymbol{\alpha}_{N+1}\cdot\boldsymbol{\varphi}\right)} d^2x.$$

Typically this perturbation breaks down all the $W-$algebra symmetry down to Poincaré symmetry. However, there is a special class of perturbations, called the integrable ones, which survive an infinite symmetry of the original theory in a very non-trivial way (Zamolodchikov 1989). Namely, one can argue there are infinitely many mutually commuting local Integrals of Motion $\mathbf{I}_s^{\lambda}$ and $\bar{\mathbf{I}}_s^{\lambda}$ which are perturbative in λ

$$\mathbf{I}_s^{\lambda} = \mathbf{I}_s + O(\lambda), \qquad \bar{\mathbf{I}}_s^{\lambda} = \bar{\mathbf{I}}_s + O(\lambda),$$

where $(\mathbf{I}_s, \bar{\mathbf{I}}_s)$ are defined in CFT.

Thus any integrable perturbation inherits a distinguishable set of local IM's $\mathbf{I}_s$ in conformal field theory. The seminal program devoted to calculation of simultaneous spectra of $\mathbf{I}_s$ has been initiated by Bazhanov, Lukyanov and Zamolodchikov in 1994.

We use an alternative approach, based on affine Yangian symmetry. We consider the case of $\mathfrak{sl}(n)$ symmetry

$$S=\int\left(\frac{1}{8\pi}\Big(\partial_\mu\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi}\Big)+\Lambda\sum_{k=1}^{n-1}e^{b(\varphi_{k+1}-\varphi_k)}+\Lambda e^{b(\varphi_1-\varphi_n)}\right)\,d^2x.$$

With the last term dropped, we have the conformal field theory, whose symmetry algebra can be described by quantum Miura-Gelfand-Dikii transformation

$$\Big(Q\partial-\partial\varphi_n\Big)\Big(Q\partial-\partial\varphi_{n-1}\Big)\dots\Big(Q\partial-\partial\varphi_2\Big)\Big(Q\partial-\partial\varphi_1\Big)= \\ =(Q\partial)^n+\sum_{k=1}^{n}W^{(k)}(z)(Q\partial)^{n-k},$$

where $Q=b+b^{-1}$.

In fact, one can drop any other exponent, leading to different, but isomorphic $W-$algebra. For example, dropping the term $e^{b(\varphi_2-\varphi_1)}$, one has different formula

$$\Big(Q\partial-\partial\varphi_1\Big)\Big(Q\partial-\partial\varphi_n\Big)\ldots\Big(Q\partial-\partial\varphi_3\Big)\Big(Q\partial-\partial\varphi_2\Big)= \\ =(Q\partial)^n+\sum_{k=1}^{n}\tilde{W}^{(k)}(z)(Q\partial)^{n-k}.$$

By symmetry arguments, it is clear that local Integrals of Motion $\boldsymbol{I}_s$ should belong to the intersection of these two $W-$algebras. In particular, one can check that

$$\mathbf{I}_1=-\frac{1}{2\pi}\int\left[\sum_{i<j}^{n}(\boldsymbol{h}_i\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_j\cdot\partial\boldsymbol{\varphi})\right]dx,$$

$$\mathbf{I}_2=\frac{1}{2\pi}\int\left[\sum_{i<j<k}^{n}(\boldsymbol{h}_i\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_j\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_k\cdot\partial\boldsymbol{\varphi})+Q\sum_{i<j}(\boldsymbol{h}_i\cdot\partial\boldsymbol{\varphi})(\boldsymbol{h}_j\cdot\partial^2\boldsymbol{\varphi})\right]dx,$$

. . . .

where $\boldsymbol{h}_i=\boldsymbol{e}_i-\frac{1}{n}\sum_{k=1}\boldsymbol{e}_k$ indeed satisfy this requirement.

This point of view that IM's should belong to intersection of two W-algebras automatically implies that the intertwining operator T_1

$$T_1\tilde{W}^{(k)}(z) = W^{(k)}(z)T_1,$$

will be itself an Integral of Motion. The operator T_1 will be primarily important for us.

Actually it is natural to define more operators, which will map between different $W-$algebras corresponding to different permutations of factors in Miura formula. The Maulik-Okounkov $R-$matrix corresponds to elementary transposition

$$\mathcal{R}_{i,j}\Big(Q\partial - \partial\varphi_i\Big)\Big(Q\partial - \partial\varphi_j\Big) = \Big(Q\partial - \partial\varphi_j\Big)\Big(Q\partial - \partial\varphi_i\Big)\mathcal{R}_{i,j},$$

while the operator T_1 corresponds to the long cycle permutation

$$T_1 = \mathcal{R}_{1,2}\mathcal{R}_{1,3}\ldots\mathcal{R}_{1,n-1}\mathcal{R}_{1,n}.$$

The operator $\mathcal{R}_{i,j}$ acts in the tensor product of two Fock representations of Heisenberg algebra with the highest weight parameters u_i and u_j

$$\mathcal{F}_{u_i} \otimes \mathcal{F}_{u_j} \xrightarrow{\mathcal{R}_{i,j}} \mathcal{F}_{u_i} \otimes \mathcal{F}_{u_j}$$

and its matrix depends on difference $u_i - u_j$. Then it follows immediately from the definition that $\mathcal{R}_{i,j}(u_i - u_j)$ satisfies the Yang-Baxter equation

$$\begin{aligned}\mathcal{R}_{1,2}(u_1 - u_2)\mathcal{R}_{1,3}(u_1 - u_3)\mathcal{R}_{2,3}(u_2 - u_3) = \\ = \mathcal{R}_{2,3}(u_2 - u_3)\mathcal{R}_{1,3}(u_1 - u_3)\mathcal{R}_{1,2}(u_1 - u_2),\end{aligned}$$

and hence the whole machinery of quantum inverse scattering method can be applied. In particular, one can construct a family of commuting transfer-matrices on $n-$sites

$$\mathbf{T}(u) = \mathsf{Tr}'\Big(\mathcal{R}_{0,1}(u-u_1)\mathcal{R}_{0,2}(u-u_2)\dots\mathcal{R}_{0,n-1}(u-u_{n-1})\mathcal{R}_{0,n}(u-u_n)\Big)\Big|_{\mathcal{F}_u}.$$

At $u=u_1$ one has $\mathcal{R}_{0,1}=\mathcal{P}_{0,1}$ a permutation operator and hence

$$\mathbf{T}(u_1)=\mathcal{R}_{1,2}\mathcal{R}_{1,3}\dots\mathcal{R}_{1,n-1}\mathcal{R}_{1,n}=T_1,$$

which implies that $\mathbf{T}(u)$ commutes with local Integrals of Motion $\mathbf{I}_s$ and can be taken as a generating function.

The notation Tr′ corresponds to certain regularization of the trace, which goes through the introduction of a twist parameter q

$$\mathrm{Tr}'(\dots)\overset{\text{def}}{=}\lim_{q\to 1}\frac{1}{\chi(q)}\,\mathrm{Tr}\left(q^{L_0^{(0)}}\dots\right),\quad \text{where}\quad \chi(q)=\prod_{k=1}^{\infty}\frac{1}{1-q^k}$$

and $L_0^{(0)}=\sum_{k>0}a_{-k}^{(0)}a_k^{(0)}$ is the level operator in auxiliary space $\mathcal{F}_u$. Remarkably, the introduction of the twist parameter does not spoil the integrability, that is the twist deformed transfer-matrices

$$\mathbf{T}_q(u)=\mathrm{Tr}\Big(q^{L_0^{(0)}}\mathcal{R}_{0,1}(u-u_1)\mathcal{R}_{0,2}(u-u_2)\dots\mathcal{R}_{0,n-1}(u-u_{n-1})\mathcal{R}_{0,n}(u-u_n)\Big)\Big|_{\mathcal{F}_u},$$

still commute.

On the level of local Integrals of Motion this deformation corresponds to the non-local deformation $\mathbf{I}_s \to \mathbf{I}_s(q)$ called quantum ILW_n (Intermediate Long Wave) integrable system. In particular

$$\mathbf{I}_1(q) = \frac{1}{2\pi}\int \left[\frac{1}{2}\sum_{k=1}^{n}(\partial\varphi_k)^2\right] dx,$$

$$\mathbf{I}_2(q) = \frac{1}{2\pi}\int \left[\frac{1}{3}\sum_{k=1}^{n}(\partial\varphi_k)^3 + Q\left(\frac{i}{2}\sum_{i,j}\partial\varphi_i D\partial\varphi_j + \sum_{i<j}\partial\varphi_i\partial^2\varphi_j\right)\right] dx,$$

$$\mathbf{I}_3(q) = \frac{1}{2\pi}\int \left[\frac{1}{4}\sum_{k=1}^{n}(\partial\varphi_k)^4 + \dots\right] dx,$$

. . . .

where D is the non-locality operator whose Fourier image is

$$D(k) = k\frac{1+q^k}{1-q^k}.$$

The spectrum of ILW_n integrable system is governed by finite type Bethe ansatz equations which have been conjectured by Nekrasov and Okounkov and independently by A.L.

$$q\prod_{j\neq i}\frac{(x_i-x_j-\epsilon_1)(x_i-x_j-\epsilon_2)(x_i-x_j-\epsilon_3)}{(x_i-x_j+\epsilon_1)(x_i-x_j+\epsilon_2)(x_i-x_j+\epsilon_3)}\prod_{k=1}^{n}\frac{x_i-u_k+\frac{\epsilon_3}{2}}{x_i-u_k-\frac{\epsilon_3}{2}}=1$$

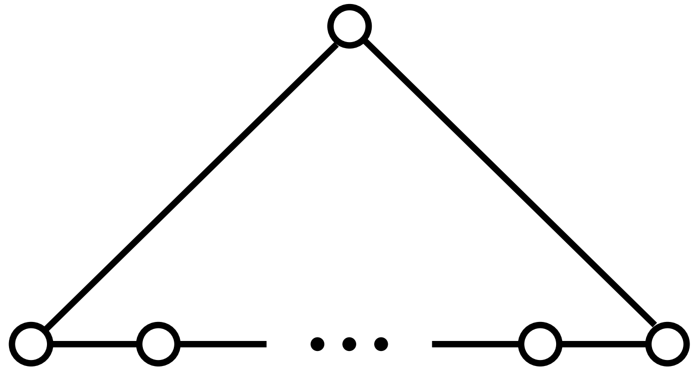
such that the eigenvalues of $\mathbf{I}_s(q)$ are symmetric polynomials in Bethe roots

$$\mathbf{I}_1(q)\sim-\frac{1}{2}\sum_{k=1}^{n}u_k^2+N,\qquad \mathbf{I}_2(q)\sim\frac{1}{3}\sum_{k=1}^{n}u_k^3-2i\sum_{j=1}^{N}x_j,\qquad \dots$$

Here we use the notations (Nekrasov)

$$b=\sqrt{\frac{\epsilon_2}{\epsilon_1}},\qquad b^{-1}=\sqrt{\frac{\epsilon_1}{\epsilon_2}}\quad\text{and}\quad \epsilon_3\overset{\text{def}}{=}-\epsilon_1-\epsilon_2.$$

The transfer-matrix $\mathbf{T}_q(u)$ corresponds to periodic boundary conditions. It is related to the affine Dynkin diagram



where each circle lives on the edge of the spin chain

$$\mathcal{F}_{u_1} \otimes \cdots \otimes \mathcal{F}_{u_n} \ ,$$

and to each pair of neighboring factors one associates the screening

$$\mathcal{F}_{u_k} \otimes \mathcal{F}_{u_{k+1}} \quad \longrightarrow \quad \oint dz\, e^{b(\varphi_k - \varphi_{k+1})} \ ,$$

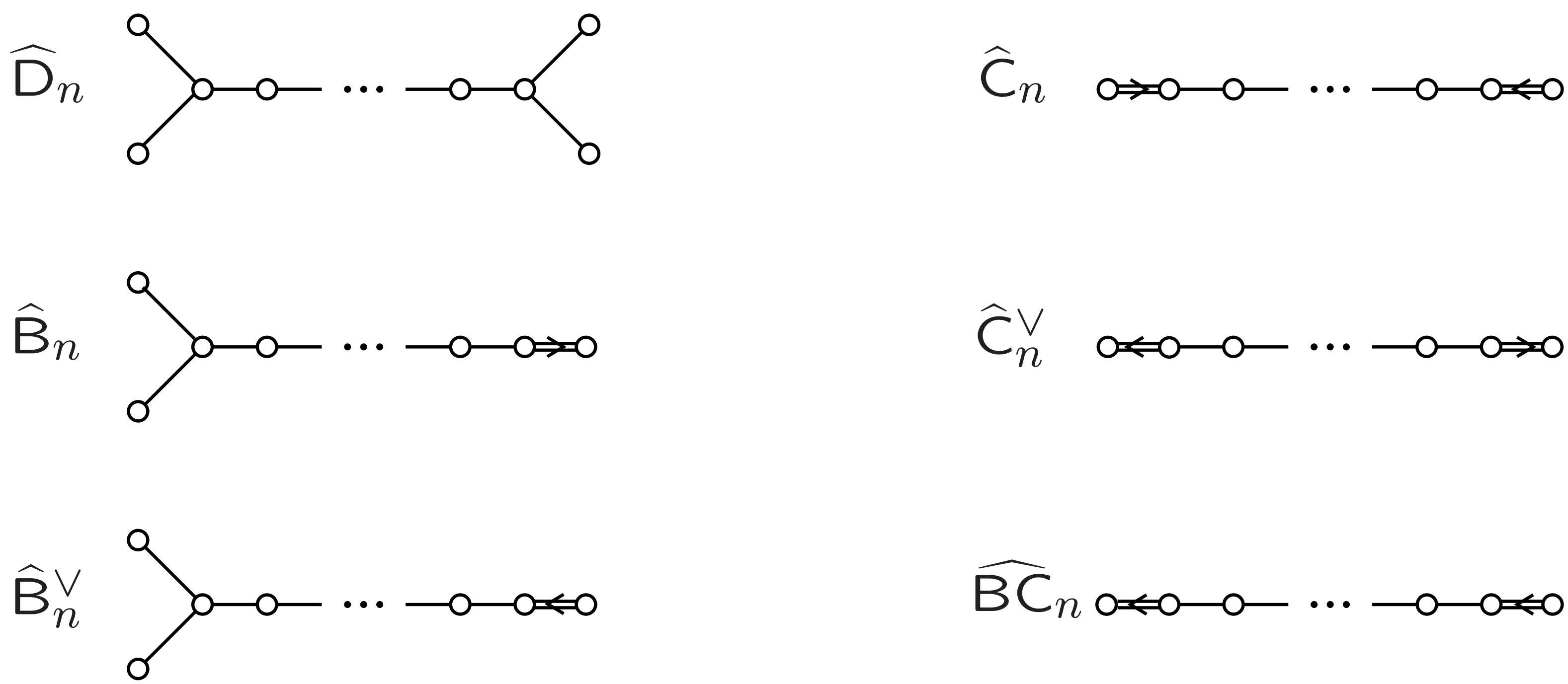
The corresponding integrable QFT is $\mathfrak{sl}(n)$ affine Toda theory

$$S = \int \left(\frac{1}{8\pi} \big(\partial_\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi} \big) + \Lambda \sum_{k=1}^{n-1} e^{b(\varphi_{k+1} - \varphi_k)} + \Lambda e^{b(\varphi_1 - \varphi_n)} \right) d^2x.$$

But one can also consider fixed b.c. In this case, according to Sklyanin, one has to find the K-operator which obeys KRKR equation

$$\mathcal{R}[\partial\varphi_1 - \partial\varphi_2]\mathcal{K}_1^\alpha \mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_2^\alpha = \mathcal{K}_2^\alpha \mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_1^\alpha \mathcal{R}[\partial\varphi_1 - \partial\varphi_2].$$

There are three solutions which correspond to affine Dynkin diagrams:



The spectrum in the boundary case is given by BAE equations

$$r^{\alpha}(x_i)r^{\beta}(x_i)A(x_i)A^{-1}(-x_i)\prod_{j\neq i}G(x_i-x_j)G^{-1}(-x_i-x_j)=1,$$

$$G(x)=\frac{(x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3)}{(x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3)},\qquad A(x)=\prod_{k=1}^{n}\frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}},$$

$$r^{\alpha}(x)=-\frac{x+\epsilon_\alpha/2}{x-\epsilon_\alpha/2}.$$

where $\alpha=1,2,3$ corresponds to three boundary conditions

$$(\boldsymbol{\alpha}_0\cdot\boldsymbol{\varphi})=\begin{cases}-\varphi_1\\ -2\varphi_1\\ -\varphi_1-\varphi_2\end{cases}\qquad (\boldsymbol{\alpha}_r\cdot\boldsymbol{\varphi})=\varphi_r-\varphi_{r+1},\quad (\boldsymbol{\alpha}_n\cdot\boldsymbol{\varphi})=\begin{cases}\varphi_n\\ 2\varphi_n\\ \varphi_{n-1}+\varphi_n\end{cases}$$

In particular, the spectrum of $\mathbf{I}_3$ is given by

$$\mathbf{I}_3\sim\mathbf{I}_3^{\mathsf{vac}}+\left(4N-4\sum_{k=1}^{n}\frac{u_k^2}{\epsilon_1\epsilon_2}+\frac{\epsilon_1^2+\epsilon_2^2}{3\epsilon_1\epsilon_2}\left(2n-\frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\right)N+$$
$$+\frac{4}{\epsilon_1\epsilon_2}\left(2n-\frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\sum_{k=1}^{N}x_k^2,$$

R-matrix

It is clear from the definition that $\mathcal{R}_{i,j}$ trivially commutes with the center of mass field $\varphi_i + \varphi_j$, that is

$$\mathcal{R}_{i,j} = \mathcal{R}\Big|_{J \to \frac{\partial\varphi_i - \partial\varphi_j}{2}},$$

where $\mathcal{R}$ is the Liouville reflection operator for the $U(1)$ current algebra

$$J(z)J(w) = \frac{1}{2(z-w)^2} + \dots$$

which is defined as

$$\mathcal{R}(-J^2 + Q\partial J) = (-J^2 - Q\partial J)\mathcal{R}. \qquad (*)$$

This relation can be used for calculation of $\mathcal{R}$. Consider highest weight representation generated by the negative mode operators a_{-k} from the vacuum state $|u\rangle$:

$$a_0|u\rangle = u|u\rangle, \qquad a_n|u\rangle = 0 \quad \text{for} \quad n > 0.$$

Then (*) is equivalent to the infinite set of relations

$$\mathcal{R}L^{(+)}_{-\lambda_1}\dots L^{(+)}_{-\lambda_n}|u\rangle = \mathcal{R}^{\mathsf{vac}}(u)L^{(-)}_{-\lambda_1}\dots L^{(-)}_{-\lambda_n}|u\rangle,$$

where $L_n^{(\pm)}$ are the components of $T^{(\pm)} = -J^2 \pm Q\partial J$

$$L_n^{(\pm)} = \sum_{k\neq 0,n} a_k a_{n-k} + (2a_0 \pm inQ)a_n, \qquad L_0^{(+)} = L_0^{(-)} = \frac{Q^2}{4} + a_0^2 + 2\sum_{k>0} a_{-k}a_k.$$

and $\mathcal{R}^{\mathsf{vac}}(u)$ is an eigenvalue for the vacuum state. One can compute the matrix of $\mathcal{R}$. For example at the level 1 one has

$$\mathcal{R}L^{(+)}_{-1}|u\rangle = L^{(-)}_{-1}|u\rangle \implies \mathcal{R}a_{-1}|u\rangle = \frac{2u+iQ}{2u-iQ}a_{-1}|u\rangle.$$

Similarly, at the level 2 one obtains

$$\mathcal{R}a_{-2}|u\rangle = \frac{\left(\left(8u^3 + 2u(3Q^2-1) - iQ(2Q^2+1)\right)a_{-2} - 8iQua_{-1}^2\right)|u\rangle}{(2u-iQ)(2u-iQ-ib)(2u-iQ-ib^{-1})},$$

$$\mathcal{R}a_{-1}^2|u\rangle = \frac{\left(-4iQua_{-2} + \left(8u^3 + 2u(3Q^2-1) + iQ(2Q^2+1)\right)a_{-1}^2\right)|u\rangle}{(2u-iQ)(2u-iQ-ib)(2u-iQ-ib^{-1})}.$$

Commutation relations of the Yang-Baxter algebra

The Maulik-Okounkov R-matrix defines the Yang-Baxter algebra in a standard way

$$\mathcal{R}_{ij}(u-v)\mathcal{L}_i(u)\mathcal{L}_j(v) = \mathcal{L}_j(v)\mathcal{L}_i(u)\mathcal{R}_{ij}(u-v).$$

Here $\mathcal{L}_i(u)$ is treated as an operator in some quantum space, a tensor product of n Fock spaces in our case, and as a matrix in auxiliary Fock space $\mathcal{F}_u$. This algebra becomes an infinite set of quadratic relations between the matrix elements labeled by two partitions

$$\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u) \overset{\text{def}}{=} \langle u|a_{\boldsymbol{\lambda}}\mathcal{L}(u)a_{-\boldsymbol{\mu}}|u\rangle \quad \text{where} \quad a_{-\boldsymbol{\mu}}|u\rangle = a_{-\mu_1}a_{-\mu_2}\ldots|u\rangle.$$

We introduce three basic currents of degree 0, 1 and -1

$$h(u) \stackrel{\text{def}}{=} \mathcal{L}_{\varnothing,\varnothing}(u), \quad e(u) \stackrel{\text{def}}{=} h^{-1}(u) \cdot \mathcal{L}_{\varnothing,\square}(u), \quad f(u) \stackrel{\text{def}}{=} \mathcal{L}_{\square,\varnothing}(u) \cdot h^{-1}(u),$$

as well as an auxiliary current

$$\psi(u) \stackrel{\text{def}}{=} \Big(\mathcal{L}_{\square,\square}(u-Q) - \mathcal{L}_{\varnothing,\square}(u-Q)h^{-1}(u-Q)\mathcal{L}_{\square,\varnothing}(u-Q)\Big)h^{-1}(u-Q)$$

As follows from the definition of R they admit large u expansion

$$h(u) = 1 + \frac{h_0}{u} + \frac{h_1}{u^2} + \ldots, \quad e(u) = \frac{e_0}{u} + \frac{e_1}{u^2} + \ldots,$$
$$f(u) = \frac{f_0}{u} + \frac{f_1}{u^2} + \ldots, \quad \psi(u) = 1 + \frac{\psi_0}{u} + \frac{\psi_1}{u^2} + \ldots$$

It proves convenient to introduce higher currents labeled by $3D$ partitions. In particular, on level 2 one has three $e_{\boldsymbol{\lambda}}(u)$ currents

$$e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(u) = \frac{ibQ}{(b^2-1)(b^2+2)} h^{-1}(u) \left(\mathcal{L}_{\varnothing,\square\square}(u) - ib\mathcal{L}_{\varnothing,\begin{smallmatrix}\square\\\square\end{smallmatrix}}(u) \right),$$

$$e_{\square\square}(u) = \frac{ib^{-1}Q}{(b^{-2}-1)(b^{-2}+2)} h^{-1}(u) \left(\mathcal{L}_{\varnothing,\square\square}(u) - ib^{-1}\mathcal{L}_{\varnothing,\begin{smallmatrix}\square\\\square\end{smallmatrix}}(u) \right),$$

$$e_{\square\!\diagup\!\square}(u) = Q\left[b e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(u) + b^{-1} e_{\square\square}(u) - e^2(u) \right].$$

and similarly for $f_{\boldsymbol{\lambda}}(u)$. Then we have:

h, e, f, ψ relations

$$[h(u), \psi(v)] = 0, \quad [\psi(u), \psi(v)] = 0, \quad [h(u), h(v)] = 0,$$

$$(u - v - \epsilon_3)h(u)e(v) = (u - v)e(v)h(u) - \epsilon_3 h(u)e(u),$$

$$(u - v - \epsilon_3)f(v)h(u) = (u - v)h(u)f(v) - \epsilon_3 f(u)h(u),$$

$$(u - v)[e(u), f(v)] = \psi(u) - \psi(v),$$

$ee, \ ff$ relations

$$g(u-v)\left[e(u)e(v) - \frac{e_{\square\square}(v)}{u - v + \epsilon_1} - \frac{e_{\substack{\square\\\square}}(v)}{u - v + \epsilon_2} - \frac{e_{\square\!\!\square}(v)}{u - v + \epsilon_3}\right] =$$

$$= \bar{g}(u-v)\left[e(v)e(u) - \frac{e_{\square\square}(u)}{u - v - \epsilon_1} - \frac{e_{\substack{\square\\\square}}(u)}{u - v - \epsilon_2} - \frac{e_{\square\!\!\square}(u)}{u - v - \epsilon_3}\right],$$

$$\bar{g}(u-v)\left[f(u)f(v) - \frac{f_{\square\square}(v)}{u - v - \epsilon_1} - \frac{f_{\substack{\square\\\square}}(v)}{u - v - \epsilon_2} - \frac{f_{\square\!\!\square}(v)}{u - v - \epsilon_3}\right] =$$

$$= g(u-v)\left[f(v)f(u) - \frac{f_{\square\square}(u)}{u - v + \epsilon_1} - \frac{f_{\substack{\square\\\square}}(u)}{u - v + \epsilon_2} - \frac{f_{\square\!\!\square}(u)}{u - v + \epsilon_3}\right],$$

where

$$g(x) \overset{\text{def}}{=} (x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3), \quad \bar{g}(x) \overset{\text{def}}{=} (x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3).$$

and Serre relations

$$\sum_{\sigma\in\mathbb{S}_3} (u_{\sigma_1} - 2u_{\sigma_2} + u_{\sigma_3}) e(u_{\sigma_1}) e(u_{\sigma_2}) e(u_{\sigma_3}) + \\ + \sum_{\sigma\in\mathbb{S}_3} [e(u_{\sigma_1}), e_{\square\!\!\!\square}(u_{\sigma_2}) + e_{\square\square}(u_{\sigma_2}) + e_{\substack{\square\\\square}}(u_{\sigma_2})] = 0,$$

and

$$\sum_{\sigma\in\mathbb{S}_3} (u_{\sigma_1} - 2u_{\sigma_2} + u_{\sigma_3}) f(u_{\sigma_1}) f(u_{\sigma_2}) f(u_{\sigma_3}) + \\ + \sum_{\sigma\in\mathbb{S}_3} [f(u_{\sigma_1}), f_{\square\!\!\!\square}(u_{\sigma_2}) + f_{\square\square}(u_{\sigma_2}) + f_{\substack{\square\\\square}}(u_{\sigma_2})] = 0.$$

Zero twist integrable system

Suppose, one has an eigenvector of $h(u)$

$$h(u)|\Lambda\rangle = h_\Lambda(u)|\Lambda\rangle,$$

then one can try to create new states by repetitive application of $e(v)$. From commutation relations one finds that

$$h(u)e(v)|\Lambda\rangle = \frac{u-v}{u-v-\epsilon_3} h_\Lambda(u)e(v)|\Lambda\rangle - \frac{\epsilon_3}{u-v-\epsilon_3} L_{\varnothing,\square}(u)|\Lambda\rangle,$$

and hence in general $e(v)|\Lambda\rangle$ is not an eigenvector of $h(u)$. However if $e(v)|\Lambda\rangle$ develops a singularity at some value $v=x$, typically a pole, then the second term is negligible and we have a new eigenvector

$$|\tilde{\Lambda}\rangle = \frac{1}{2\pi i}\oint_{\mathcal{C}_x} e(v)|\Lambda\rangle dv, \qquad h(u)|\tilde{\Lambda}\rangle = \frac{(u-x)}{(u-x-\epsilon_3)} h_\Lambda(u)|\tilde{\Lambda}\rangle$$

Using this property, one can generate any eigenvector from the vacuum state by successive application of $e(u)$. We note that the operators $e(u)$ do not commute. However we have

$$\oint_{\mathcal{C}_y} dv \oint_{\mathcal{C}_x} du\, e(u)e(v)|\Lambda\rangle = \prod_{\alpha=1}^{3} \frac{(x-y-\epsilon_\alpha)}{(x-y+\epsilon_\alpha)} \oint_{\mathcal{C}_y} dv \oint_{\mathcal{C}_x} du\, e(v)e(u)|\Lambda\rangle$$

provided that x and y are *simple* poles and that $y \neq x + \epsilon_\alpha$. Consider the tensor product of n Fock modules generated from the vacuum state $|\varnothing\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$

$$\mathcal{F}_{x_1} \otimes \cdots \otimes \mathcal{F}_{x_n} = \mathrm{span}\{a^{(1)}_{-\boldsymbol{\lambda}^{(1)}} \ldots a^{(n)}_{-\boldsymbol{\lambda}^{(n)}}|\varnothing\rangle : \boldsymbol{\lambda}^{(k)} = \lambda^{(k)}_1 \geq \lambda^{(k)}_2 \geq \ldots\}.$$

Our normalizations of $h(u)$ and $\psi(u)$ imply that

$$h(u)|\varnothing\rangle = |\varnothing\rangle, \quad \psi(u)|\varnothing\rangle = \prod_{k=1}^{n} \frac{u - x_k + \epsilon_3}{u - x_k}|\varnothing\rangle.$$

Moreover the vacuum state is annihilated by $f(u)$

$$f(u)|\varnothing\rangle = 0,$$

while the new states are generated by the modes of $e(u)$. Moreover the vacuum state is annihilated by $f(u)$, while the new states are generated by the modes of $e(u)$. The eigenfunctions of $h(u)$ provide a basis $|\vec{\boldsymbol{\lambda}}\rangle$

$$|\vec{\boldsymbol{\lambda}}\rangle \sim \oint_{\mathcal{C}_N} du_N \cdots \oint_{\mathcal{C}_1} du_1 \, e(u_N) \ldots e(u_1)|\varnothing\rangle, \qquad N = |\vec{\boldsymbol{\lambda}}| = \sum_{k=1}^{n} |\boldsymbol{\lambda}^{(k)}|,$$

The contours go counterclockwise around simple poles located at the contents of Young diagrams in $\vec{\boldsymbol{\lambda}}$

$$c_\square = x_k - (i-1)\epsilon_1 - (j-1)\epsilon_2.$$

ILW Integrals of Motion and Bethe ansatz

Consider the monodromy matrix on n sites $\mathbf{T}_q(u)$. One can easily see that $\mathbf{T}_q(u)$ admits the following large u expansion

$$\mathbf{T}_q(u) = \Lambda(u,q)\,\mathrm{exp}\left(\frac{1}{u}\mathbf{I}_1 + \frac{1}{u^2}\mathbf{I}_2 + \dots\right),$$

where $\Lambda(u,q)$ is a normalization factor and $\mathbf{I}_1$ and $\mathbf{I}_2$ are the first ILW$_n$ Integrals of Motion. Among other Integrals of Motion there is a particular one called KZ integral

$$T_1 \overset{\mathrm{def}}{=} \mathbf{T}_q(u_1).$$

Using the fact that $\mathcal{R}_{0,1}(0) = \mathcal{P}_{0,1}$, one finds

$$T_1 = q^{L_0^{(1)}}\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\dots\mathcal{R}_{1,n}(u_1-u_n).$$

We take the tensor product of $n+N$ Fock spaces

$$\underbrace{\mathcal{F}_{u_1}\otimes\cdots\otimes\mathcal{F}_{u_n}}_{\text{quantum space}}\otimes\underbrace{\mathcal{F}_{x_1}\otimes\cdots\otimes\mathcal{F}_{x_N}}_{\text{auxiliary space}}$$

Consider the special state in the auxiliary space

$$|\chi\rangle_{\boldsymbol{x}} \overset{\text{def}}{=} |\underbrace{\square,\dots,\square}_{N}\rangle \sim \oint_{\mathcal{C}_N} dz_N \cdots \oint_{\mathcal{C}_1} dz_1\, e(z_N)\dots e(z_1)|\varnothing\rangle_{\boldsymbol{x}},$$

where the contour $\mathcal{C}_k$ encircles the point x_k.

$$h(u)|\chi\rangle_{\boldsymbol{x}} = \prod_{k=1}^{N} \frac{u-x_k}{u-x_k-\epsilon_3}|\chi\rangle_{\boldsymbol{x}}.$$

and (here $S(x)=\frac{(x+\epsilon_1)(x+\epsilon_2)}{x(x+\epsilon_3)}$)

$${}_{\boldsymbol{x}}\langle\varnothing|f(z)\dots f(z_1)|\chi\rangle_{\boldsymbol{x}} = \mathrm{Sym}_{\boldsymbol{x}}\left(\prod_{a=1}^{N}\frac{1}{z_a-x_a}\prod_{a<b}S(x_a-x_b)\right),$$

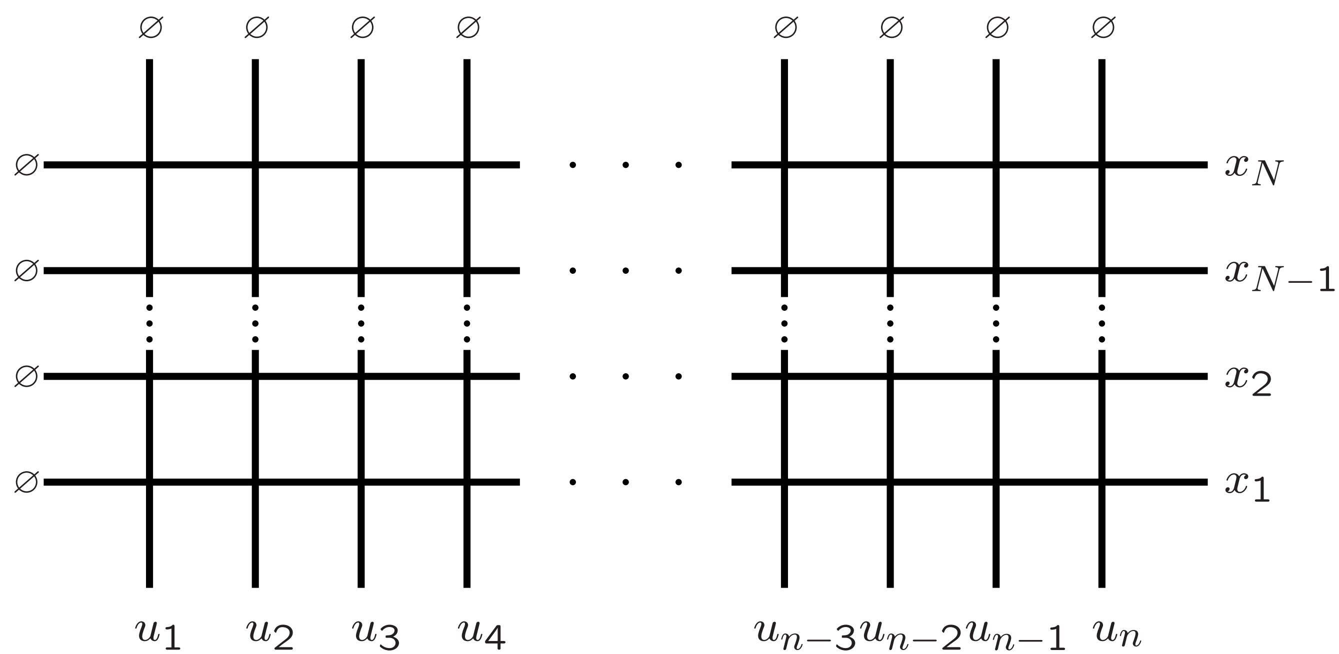
Now we define the off-shell Bethe vector as

$$|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} \overset{\text{def}}{=} {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}},$$

where

$$\mathcal{R}(\boldsymbol{x},\boldsymbol{u}) = \mathcal{R}_{x_1u_1}\dots\mathcal{R}_{x_Nu_1}\dots\mathcal{R}_{x_1u_n}\dots\mathcal{R}_{x_Nu_n}.$$

The off-shell Bethe vector $|\Psi(\boldsymbol{x})\rangle$ can be represented by the following picture

$|B(\boldsymbol{x})\rangle =$  $|\chi\rangle_{\boldsymbol{x}}$

Consider the matrix element between $|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}$ and generic state

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) \stackrel{\text{def}}{=} {}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} = {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}},$$

It can be expressed through $h(u)$ and $f(z)$ via contour integral

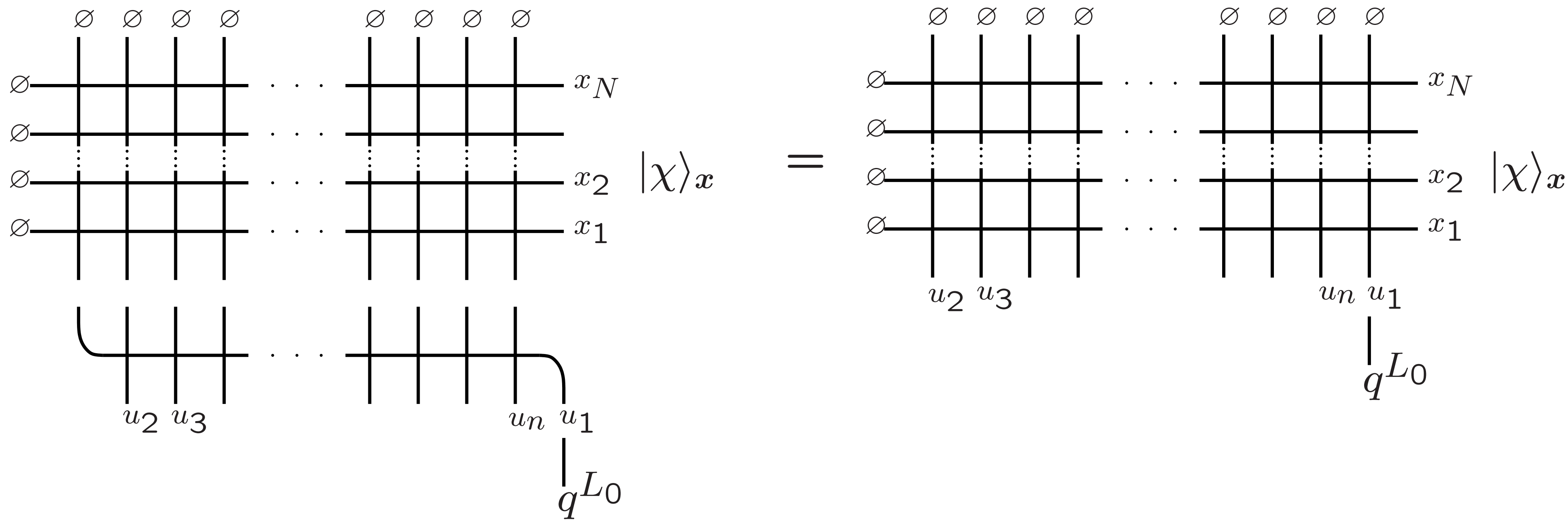
$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \frac{1}{(2\pi i)^N}\times$$

$$\times\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\ {}_{\boldsymbol{x}}\langle\varnothing|h(u_1)\underbrace{f(z^{(1)}_1)f(z^{(1)}_2)\dots}_{|\boldsymbol{\lambda}^{(1)}|}h(u_2)\underbrace{f(z^{(2)}_1)f(z^{(2)}_2)\dots}_{|\boldsymbol{\lambda}^{(2)}|}\ \dots\ h(u_n)\underbrace{f(z^{(}_1}$$

where

$$F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u}) = \prod_{k=1}^{n} F_{\boldsymbol{\lambda}^{(k)}}\left(z^{(k)}_1,\dots,z^{(k)}_{|\boldsymbol{\lambda}^{(k)}|}\middle|u_k\right).$$

The action of the KZ Integral of Motion on off-shell Bethe vector $|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}$ is very simple and can be explained by the following picture



Projecting this equation on arbitrary state, one obtains

$$
{}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|T_1|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} =
$$

$$
= q^{|\boldsymbol{\lambda}^{(1)}|}{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(2)},\varnothing}(u_2)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)|\chi\rangle_{\boldsymbol{x}}
$$

If we require that $|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}$ is an eigenstate for T_1 we have to demand

$$q^{|\boldsymbol{\lambda}^{(1)}|}\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(2)},\varnothing}(u_2)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)|\chi\rangle_{\boldsymbol{x}} = \\ = T_1(\boldsymbol{u})\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}},$$

which should hold for any set of partitions $\vec{\boldsymbol{\lambda}}$. The eigenvalue $T_1(\boldsymbol{u})$ is

$$T_1(\boldsymbol{u}) = \prod_{k=1}^{N}\frac{x_k-u_1}{x_k-u_1+\epsilon_3}.$$

For generic $\vec{\boldsymbol{\lambda}}$ the eigenstate equation implies the integral identity

$$q^{|\boldsymbol{\lambda}^{(1)}|}\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\ {}_{\boldsymbol{x}}\langle\varnothing|h(u_2)\underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\ \dots\ h(u_n)\underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}h(u_1)\underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}|\chi\rangle_{\boldsymbol{x}}$$
$$= T_1(\boldsymbol{u})\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\ {}_{\boldsymbol{x}}\langle\varnothing|\,h(u_1)\underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}h(u_2)\underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\ \dots\ h(u_n)\underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}$$

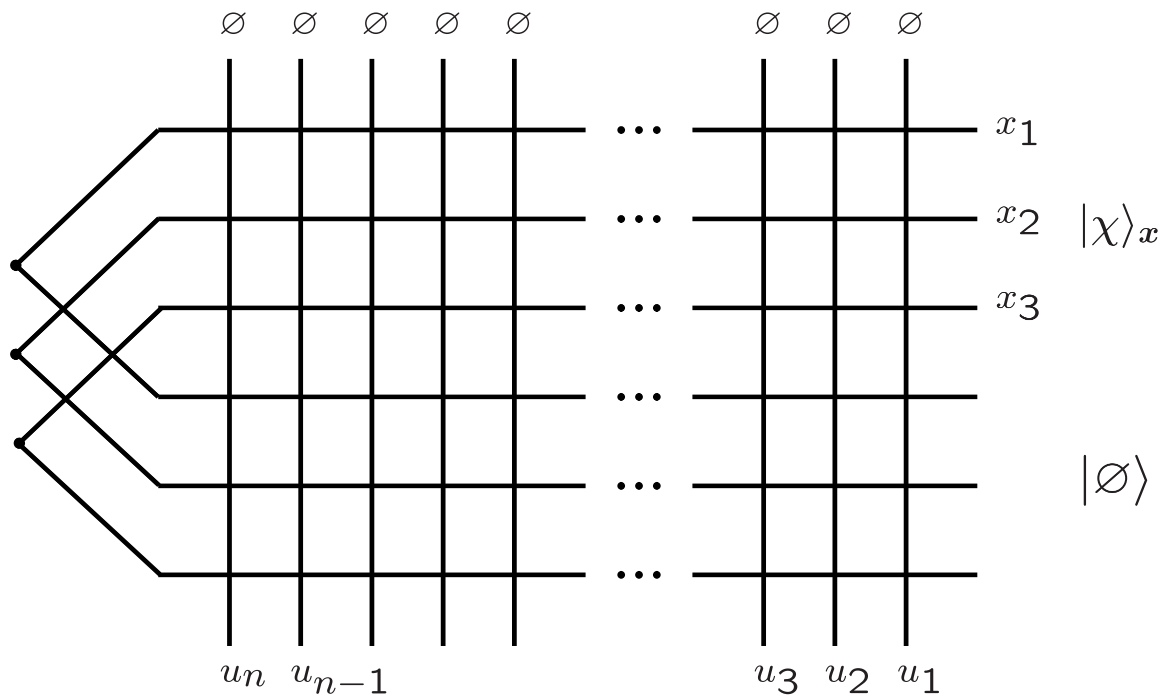
which holds provided that $\boldsymbol{x}$ obeys Bethe ansatz equations

$$q\prod_{j\neq i}\prod_{\alpha=1}^{3}\frac{x_i-x_j-\epsilon_\alpha}{x_i-x_j+\epsilon_\alpha}\prod_{k=1}^{n}\frac{x_i-u_k+\epsilon_3}{x_i-u_k} = 1 \quad \text{for all} \quad i=1,\dots,N.$$

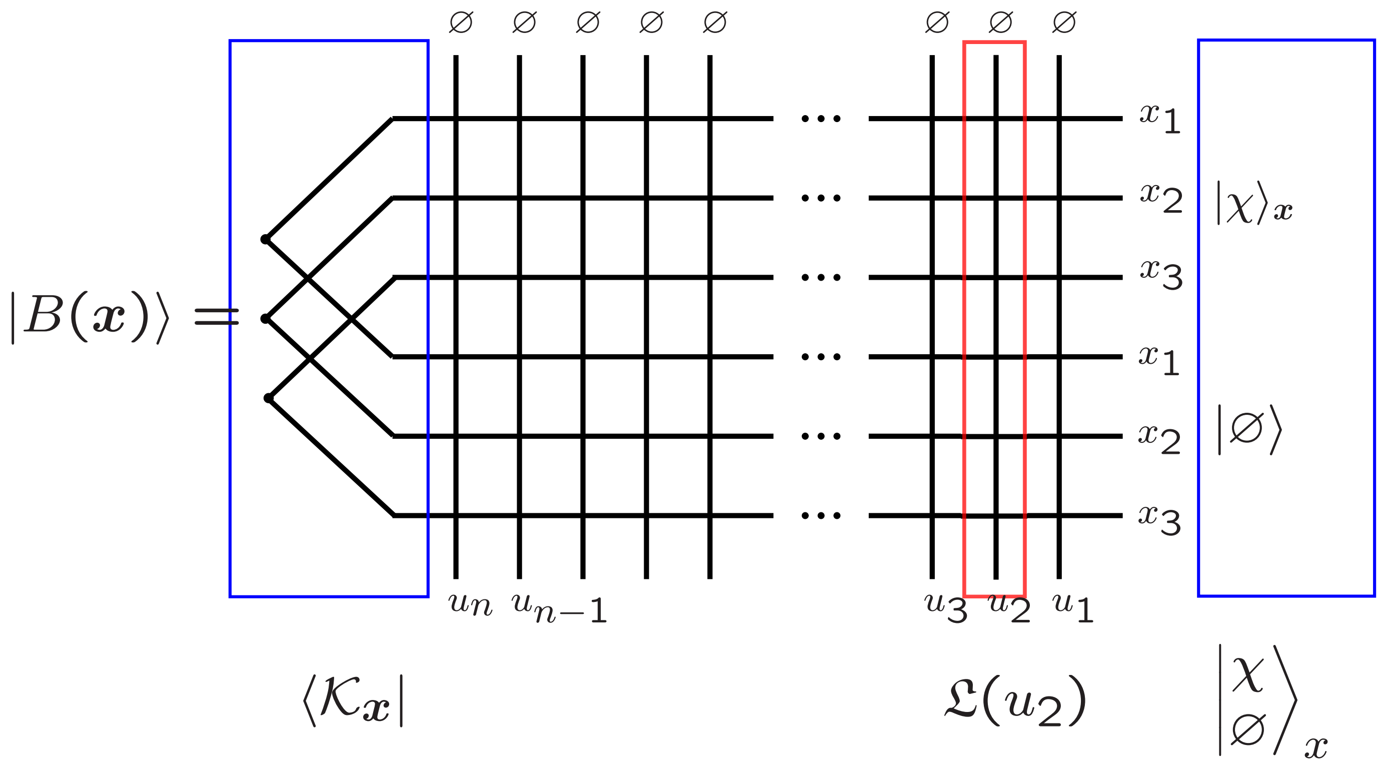
Boundary Bethe ansatz

In the boundary case we found the following representation for the off-shell Bethe vector

$|B(\boldsymbol{x})\rangle =$



The formula for the off-shell Bethe vector can be revised. One observes that the definition of $|B(\boldsymbol{x})\rangle$ can be interpreted as a product of some $L-$operators $\mathfrak{L}(u_n)\ldots\mathfrak{L}(u_1)$ sandwiched between $\langle\mathcal{K}_{\boldsymbol{x}}|$ and $\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_x$



Then one can proceed in exactly the same way as in the periodic case.

$\mathbf{Y}(\widehat{\mathfrak{gl}}(2))$

The corresponding $R-$matrix should be searched in the NSR algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\widehat{c}}{8}(m^3 - m)\delta_{m,-n},$$
$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r},$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{\widehat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}.$$

We define the operator $\mathcal{R}$ as (we assume that $\mathcal{R}_{\text{vac}}(u) = 1$)

$$\mathcal{R} L^{(+)}_{-\boldsymbol{\lambda}} G^{(+)}_{-\boldsymbol{r}} |u\rangle = L^{(-)}_{-\boldsymbol{\lambda}} G^{(-)}_{-\boldsymbol{r}} |u\rangle,$$

where

$$G^{\pm}_r = \sum_{k\neq 0} a_k \psi_{r-k} + (a_0 \pm irQ)\,\psi_r,$$
$$L^{\pm}_n = \frac{1}{2}\sum_{k\neq 0,n} a_k a_{n-k} + \frac{1}{2}\sum_r r\psi_{n-r}\psi_r + \left(a_0 \pm \frac{inQ}{2}\right) a_n.$$

One can compute the matrix of $\mathcal{R}$. For example at level $\frac{1}{2}$ one has

$$\mathcal{R}G^{+}_{-\frac{1}{2}}|u\rangle = G^{-}_{-\frac{1}{2}}|u\rangle \implies \mathcal{R}\psi_{-\frac{1}{2}}|u\rangle = \frac{2u+iQ}{2u-iQ}\psi_{-\frac{1}{2}}|u\rangle$$

On level 1

$$\mathcal{R}a_{-1}|u\rangle = \frac{2u+iQ}{2u-iQ}a_{-1}|u\rangle$$

On level $\frac{3}{2}$:

$$\mathcal{R}\psi_{-\frac{3}{2}}|u\rangle = \frac{\left((u-\frac{iQ}{2})^2(u+\frac{3iQ}{2})-(u+\frac{iQ}{2})\right)\psi_{-\frac{3}{2}}|u\rangle - 2iuQa_{-1}\psi_{-\frac{1}{2}}|u\rangle}{(u-\frac{iQ}{2})(u-\frac{iQ}{2}-ib)(u-\frac{iQ}{2}-ib^{-1})}$$

$$\mathcal{R}a_{-1}\psi_{-\frac{1}{2}}|u\rangle = \frac{-2iuQ\psi_{-\frac{3}{2}}|u\rangle + \left((u+\frac{iQ}{2})^2(u-\frac{3iQ}{2})-(u-\frac{iQ}{2})\right)\psi_{-\frac{1}{2}}|u\rangle}{(u-\frac{iQ}{2})(u-\frac{iQ}{2}-ib)(u-\frac{iQ}{2}-ib^{-1})}$$

etc

The $\widehat{\mathfrak{gl}}(n)_k$ has the form

$$E_{ij}(z)E_{kl}(w) = \frac{\kappa\delta_{il}\delta_{jk}}{(z-w)^2} + \frac{\delta_{jk}E_{il}(w) - \delta_{il}E_{kj}(w)}{z-w} + \text{reg}$$

The trace current $U(z) = \sum_k E_{kk}(z)$ trivially decouples so that we have the decomposition $\widehat{\mathfrak{gl}}(n)_\kappa = \mathcal{H} \otimes \widehat{\mathfrak{sl}}(n)_\kappa$, where $\mathcal{H}$ is the Heisenberg algebra.

It is well known that for $\kappa = 1$ the algebra admits free fermion representation

$$E_{ij} =: \psi_i^* \psi_j,$$

$$\psi_i^*(z)\psi_j(w) = \frac{\delta_{ij}}{z-w} + \text{reg}, \quad \psi_i^*(z)\psi_j^*(w) = \text{reg}, \quad \psi_i(z)\psi_j(w) = \text{reg}.$$

and that each free fermion can be represented as

$$\psi_k = e^{i\phi_k} \quad \psi_k^* = e^{-i\phi_k}$$

The R-matrix $\mathcal{R}_{ij}$ should be an embedding of super Liouville reflection operator $\mathcal{R}$ into

$$\widehat{\mathfrak{gl}}(2)_1 \oplus \cdots \oplus \widehat{\mathfrak{gl}}(2)_1,$$

such that it is non-trivial only in ith and jth copies of $\widehat{\mathfrak{gl}}(2)_1$. In order to do so, for each $\widehat{\mathfrak{gl}}(2)_1$ we bosonize fermions in its current matrix

$$\boldsymbol{E}^{(j)} = \begin{pmatrix} i\partial\phi_1^{(j)} & e^{i\left(\phi_1^{(j)}-\phi_2^{(j)}\right)} \\ e^{i\left(\phi_2^{(j)}-\phi_1^{(j)}\right)} & i\partial\phi_2^{(j)} \end{pmatrix},$$

and define

$$R_{ij} \overset{\text{def}}{=} \mathcal{R}[\Phi, \Psi],$$

where

$$\Phi = \frac{1}{2}\left(\phi_1^{(i)} - \phi_1^{(j)} + \phi_2^{(i)} - \phi_2^{(j)}\right),$$

$$\Psi = \frac{1}{i\sqrt{2}}\left(e^{\frac{i}{2}\left((\phi_1^{(i)}-\phi_1^{(j)})-(\phi_2^{(i)}-\phi_2^{(j)})\right)} - e^{-\frac{i}{2}\left((\phi_1^{(i)}-\phi_1^{(j)})-(\phi_2^{(i)}-\phi_2^{(j)})\right)}\right)$$

We note that from this definition it follows that $\mathcal{R}_{ij}$ automatically commutes with

$$\phi_1^{(i)} + \phi_1^{(j)}, \quad \phi_2^{(i)} + \phi_2^{(j)}$$

and

$$\chi \stackrel{\text{def}}{=} e^{\frac{i}{2}\left((\phi_1^{(i)}-\phi_1^{(j)})-(\phi_2^{(i)}-\phi_2^{(j)})\right)} + e^{-\frac{i}{2}\left((\phi_1^{(i)}-\phi_1^{(j)})-(\phi_2^{(i)}-\phi_2^{(j)})\right)}.$$

or noticing that

$$\boldsymbol{E}^{(i)} + \boldsymbol{E}^{(j)} = \begin{pmatrix} i\left(\partial\phi_1^{(i)} + \partial\phi_1^{(j)}\right) & e^{\frac{i}{2}\left((\phi_1^{(i)}+\phi_1^{(j)})-(\phi_2^{(i)}+\phi_2^{(j)})\right)}\chi \\ e^{-\frac{i}{2}\left((\phi_1^{(i)}+\phi_1^{(j)})-(\phi_2^{(i)}+\phi_2^{(j)})\right)}\chi & i\left(\partial\phi_2^{(i)} + \partial\phi_2^{(j)}\right) \end{pmatrix},$$

it means that $\mathcal{R}_{ij}$ commutes with $\boldsymbol{E}^{(i)} + \boldsymbol{E}^{(j)}$. That is the algebra is

$$\widehat{\mathfrak{gl}}(2)_2 \times \mathsf{NSR}$$

In general $\mathsf{Y}(\widehat{\mathfrak{gl}}(2)) \to \mathsf{Y}(\widehat{\mathfrak{gl}}(p))$

$$\widehat{\mathfrak{gl}}(p)_2 \times \mathcal{A}(2,p)$$

where $\mathcal{A}(2,p)$ is the chiral algebra for the coset CFT (para-Liouville CFT)

$$\frac{\widehat{\mathfrak{sl}}(2)_p \times \widehat{\mathfrak{sl}}(2)_{n-p}}{\widehat{\mathfrak{sl}}(2)_n}.$$

The basis in rep of $\widehat{\mathfrak{gl}}(2)_1$ is known to be given by colored partitions (chess partitions). The YB algebra is generated by six currents:

$$h_1(u) = \mathcal{L}_{\circ,\circ} = \langle 0|\mathcal{L}(u)|0\rangle \quad h_2(u) = \mathcal{L}_{\bullet,\bullet} = \Big\langle \frac{1}{2}\Big|\mathcal{L}(u)\Big|\frac{1}{2}\Big\rangle$$

$$e_1(u) = h_1^{-1}(u)\mathcal{L}_{\circ,\square}(u) = h_1^{-1}(u)\langle 0|\mathcal{L}(u)|1\rangle,$$
$$e_2(u) = h_2^{-1}(u)\mathcal{L}_{\bullet,\blacksquare}(u) = h_2^{-1}(u)\Big\langle \frac{1}{2}\Big|\mathcal{L}(u)\Big| -\frac{1}{2}\Big\rangle$$

$$f_1(u) = \mathcal{L}_{\square,\circ}(u)h_1^{-1}(u) = \langle 1|\mathcal{L}(u)|0\rangle h_1^{-1}(u),$$
$$f_2(u) = \mathcal{L}_{\blacksquare,\bullet}(u)h_2^{-1}(u) = \Big\langle -\frac{1}{2}\Big|\mathcal{L}(u)\Big|\frac{1}{2}\Big\rangle h_2^{-1}(u)$$

and auxiliary currents

$$\psi_1(u+Q) = \mathcal{L}_{\square,\square}(u)h_1^{-1}(u) - \mathcal{L}_{\circ,\square}(u)h_1^{-1}(u)L_{\square,\circ}(u)h_1^{-1}(u)$$
$$\psi_2(u+Q) = \mathcal{L}_{\blacksquare,\blacksquare}(u)h_1^{-1}(u) - \mathcal{L}_{\bullet,\blacksquare}(u)h_1^{-1}(u)L_{\blacksquare,\bullet}(u)h_1^{-1}(u)$$

One can find commutation relations (some of them)

$$[h_i(u), h_j(u)] = 0, \quad \forall i, j = \{1, 2\}$$

$$[h_i(u), e_j(v)] = [h_i(u), f_j(v)] = 0, \quad \forall i \neq j = \{1, 2\}$$

$$(\Delta + \epsilon_3) h_1(u) e_1(v) = \epsilon_3 \mathcal{L}_{\circ,\square}(u) + \Delta e_1(v) h_1(u),$$
$$(\Delta + \epsilon_3) h_2(u) e_2(v) = \epsilon_3 \mathcal{L}_{\bullet,\blacksquare}(u) + \Delta e_2(v) h_2(u)$$

$$(\Delta + \epsilon_3) f_1(v) h_1(u) = \epsilon_3 \mathcal{L}_{\square,\circ}(u) + \Delta h_1(u) f_1(v),$$
$$(\Delta + \epsilon_3) f_2(v) h_2(u) = \epsilon_3 \mathcal{L}_{\blacksquare,\bullet}(u) + \Delta h_2(u) f_2(v)$$

$$\frac{\Delta - \epsilon_3}{\Delta} e_i(u) e_i(v) + \frac{\epsilon_3}{\Delta} e_i(v) e_i(v) = \frac{\Delta + \epsilon_3}{\Delta} e_i(v) e_i(u) - \frac{\epsilon_3}{\Delta} e_i(u) e_i(u)$$

$$g(\Delta)\left(e_1(v)e_2(u) - \frac{e_{\blacksquare\square}(u)}{\Delta + \epsilon_1} - \frac{e_{\substack{\blacksquare\\ \square}}(u)}{\Delta + \epsilon_2}\right) = \bar{g}(\Delta)\left(e_2(u)e_1(v) - \frac{e_{\square\blacksquare}(v)}{\Delta - \epsilon_1} - \frac{e_{\substack{\square\\ \blacksquare}}(v)}{\Delta - \epsilon_2}\right)$$

where

$$\Delta = u - v, \quad g(x) = (x + \epsilon_1)(x + \epsilon_2), \quad \bar{g}(x) = (x - \epsilon_1)(x - \epsilon_2).$$

Proceeding in exactly the same way as in $\mathfrak{gl}(1)$ case we find the following Bethe ansatz equations

$$\prod_{l=1}^{n}\frac{u_l-x_i}{u_l-x_i+\epsilon_3}\prod_{j\neq i}^{N_1}\frac{x_i-x_j-\epsilon_3}{x_i-x_j+\epsilon_3}\prod_{k=1}^{N_2}\frac{(y_k-x_i+\epsilon_1)(y_k-x_i+\epsilon_2)}{(y_k-x_i-\epsilon_1)(y_k-x_i-\epsilon_2)}=qt,$$

and

$$\prod_{j=1}^{N_1}\frac{(y_i-x_j-\epsilon_1)(y_i-x_j-\epsilon_2)}{(y_i-x_j+\epsilon_1)(y_i-x_j+\epsilon_2)}\prod_{k\neq i}^{N_2}\frac{y_i-y_k-\epsilon_3}{y_i-y_k+\epsilon_3}=t^{-1}.$$

where q and t are the twist parameters for the transfer-matrix

$$T_{q,t}(u)=\mathrm{tr}\Big(q^{L_0^{(0)}}t^{h^{(0)}}\mathcal{R}_{0,n}(u-u_n)...\mathcal{R}_{0,1}(u-u_1)\Big).$$

Ongoing studies

- Formfactors of local fields
- $N = 2$ W_∞ algebra (Gaberdiel et al)
- Massive deformations