Massless finite and infinite spin representations of Poincaré group in six dimensions.

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Plan

Study of the various aspects of FT in higher dimensions recently attracts much attention due to the remarkable and sometimes even unexpected properties at classical and quantum levels. The basic space-time symmetry in relativistic models is Poincaré group. Theory of unitary irreps of Poincaré group in four dimensions was constructed in $\left[\begin{array}{c} E.Wigner\,(1939,1947), \\ V.Bargmann, E.Wigner\,(1948) \end{array}\right]$. The unitary irreps in higher dimensions and their applications were considered in many papers and reviews; see e.g. lectures [X.Bekaert, N.Boulanger (2006), hep-th/0611263]. Although the generic construction of the Poincaré group irreps in any dimension can be realized by the method of induced representations, many specific aspects important for CFT and QFT deserve a separate study. Some of such aspects can be formulated only for each concrete dimension and not for all dimensions.

In this report we construct the massless finite and infinite spin irreps of the Poincaré group in 6d Minkowski space. Some aspects of such irreps were considered earlier by [L.Mezincescu, P.Townsend (2014,2017)], however many issues, especially the infinite spin representations, were not addressed and complete analysis was not done. For us the important paper is [X.Bekaert, J.Mourad, JHEP **0601** (2006)115, hep-th/0509092]. Recently there appeared the paper [S.Kuzenko, A.Pindur, Massless particles in five and higher dimensions, Phys.Lett. B812 (2021) 136020], where the unitary massless irreps of the Poincaré group in 5d Minkowski space were constructed, some issues related to irreps in arbitrary dimensions were briefly studied and the representations of super Poincaré group were considered. In our study we mostly addressed to infinite spin representations.

Lie algebra $\mathfrak{iso}(1,5)$ and its Casimir operators

To characterize the unitary irreps of d-dimensional Poincaré group $ISO^{\uparrow}(1,d-1)$, or its covering $ISpin^{\uparrow}(1,d-1)$, we need to consider the corresponding irreps of the Lie algebra $\mathfrak{iso}(1,d-1) = i\mathrm{spin}(1,d-1)$ with generators $\{\hat{P}_n, \hat{M}^{mk}\}$ (components of momentum and angular momentum) and defining relations

$$[\hat{P}_n,\, \hat{P}_m] = 0\;, \quad [\hat{P}_n,\, \hat{M}_{mk}] = i\,(\eta_{kn}\hat{P}_m - \,\eta_{mn}\hat{P}_k)\;,$$
$$[\hat{M}_{nm},\, \hat{M}_{k\ell}] = i(\eta_{nk}\hat{M}_{m\ell} - \eta_{mk}\hat{M}_{n\ell} + \eta_{m\ell}\hat{M}_{nk} - \eta_{n\ell}\hat{M}_{mk})\;,$$

where $||\eta_{mk}|| = \mathrm{diag}(+1,-1,\dots,-1)$ – metric in $\mathbb{R}^{1,d-1}$.

The algebra $\mathfrak{iso}(1,d-1)$ has $[(d+1)/2]$ Casimir operators since the algebra $iso(1,d-1)$ is obtained by contraction from the simple Lie algebra $so(d+1,\mathbb{C})$ of rank $[(d+1)/2]$.

Thus, the Lie algebra $\mathfrak{iso}(1,5)$ of 6d Poincaré group has 3 Casimir operators.

To construct Casimirs for $\mathfrak{iso}(1,5)$ we introduce the third rank tensor W_{mnk} and the vector Υ_m which are elements of $\mathcal{U}(\mathfrak{iso}(1,5))$

$$W_{mnk} = \varepsilon_{mnklpr} P^l M^{pr} \, , \qquad \Upsilon_m = \varepsilon_{mnklpr} P^n M^{kl} M^{pr} \, .$$

Here ε_{mnklpr} form the total antisymmetric tensor with normalization $\varepsilon_{012345} = 1$ and operators W_{mnk} and Υ_m obey

$$P^m W_{mnk} = 0 \, , \quad [P_l, W_{mnk}] = 0 \, , \quad P^m \Upsilon_m = 0 \, , \quad [P_l, \Upsilon_m] = 0 \, .$$

Then we define the Casimir operators for $\mathfrak{iso}(1,5)$ as

$$C_2 := P^m P_m \, , \qquad C_4 := \tfrac{1}{24} W^{mnk} W_{mnk} \, ,$$
$$C_6 := \tfrac{1}{64} \Upsilon^m \Upsilon_m$$

which are 2nd, 4th and 6th order operators in the generators of $\mathcal{U}(\mathfrak{iso}(1,5))$, respectively.

Recall that in 4d Minkowski space any 2nd rank tensor $W_{mn} = -W_{nm}$ has the decomposition in sum of self-dual $W^{(+)}_{mn}$ and anti-self-dual $W^{(-)}_{mn}$ parts $W_{mn} = W^{(+)}_{mn} + W^{(-)}_{mn}$ and we have the decomposition $W^2 = (W^{(+)})^2 + (W^{(-)})^2$ which reflects $\mathfrak{so}(4,\mathbb{C}) = \mathfrak{s}\ell(2,\mathbb{C}) + \mathfrak{s}\ell(2,\mathbb{C})$.

In 6d Minkowski space any third rank tensor W_{mnk} also has the decomposition in sum of self-dual $W^{(+)}_{mnk}$ and anti-self-dual $W^{(-)}_{mnk}$ parts

$$W_{mnk} = W^{(+)}_{mnk} + W^{(-)}_{mnk} \, , \qquad W^{(\pm)}_{mnk} := \frac{1}{2}\left(W_{mnk} \pm \frac{1}{3!}\, \varepsilon_{mnklpr} W^{lpr} \right)$$

In opposite to 4d case, in 6d case the square of the third rank tensor W_{mnk} is the contraction of self-dual and anti-self-dual parts:

$$W^{mnk} W_{mnk} = 2 W^{(+)mnk} W^{(-)}_{mnk} \, , \qquad (1)$$

since we have $W^{(+)mnk} W^{(+)}_{mnk} = W^{(-)mnk} W^{(-)}_{mnk} \equiv 0$. For this reason, in 6d case, the square of the third rank tensor W_{mnk} produces only one Casimir operator (1) and there are no more independent invariant operators constructed from W_{mnk}.

Taking into account the expressions for operators W_{mnk} and Υ_k via generators $P_k, M_{n\ell}$, we obtain explicit form of the Casimirs C_2, C_4, C_6:

$$C_2 = P^m P_m \,, \qquad C_4 = \Pi^m \Pi_m - \frac{1}{2} M^{mn} M_{mn} \, C_2 \,, \tag{2}$$

$$\begin{aligned} C_6 &= -\Pi^k M_{km} \Pi_l M^{lm} + \frac{1}{2}\Big(M^{mn} M_{mn} - 8\Big) C_4 \\ &+ \frac{1}{8}\Big[M^{kl} M_{kl} \Big(M^{mn} M_{mn} - 8\Big) + 2 M^{mn} M_{nk} M^{kl} M_{lm}\Big] C_2 \,, \end{aligned} \tag{3}$$

where we introduce new vector Π with components

$$\Pi_m := P^k M_{km} = M_{km} P^k - 5i \, P_m \,, \tag{4}$$

subject to (cf. with $\mathfrak{iso}(1, d-1)$ relations)

$$[\Pi_n, \, \Pi_k] = -i \, M_{nk} \, C_2 \,, \quad [M_{mn}, \Pi_k] = i \, (\eta_{mk} \Pi_n - \eta_{nk} \Pi_m) \,. \tag{5}$$

Further we consider the massless unitary representations of $\mathfrak{iso}(1,5)$ when:

$$C_2 \equiv P^2 = P^m P_m = 0 \,.$$

Standard massless momentum reference frame

Let the algebra (2.1), (2.2) acts in the representation space $\mathcal{H}$ with basis vectors $|k\,,\sigma\rangle$, where $P_m|k\,,\sigma\rangle = k_m|k\,,\sigma\rangle$ and σ are eigenvalues of all operators which generate commuting set with P_m. We take the states $|k\,,\sigma\rangle$ for which the spectrum of momentum operators P_m form the light-cone reference frame for massless particle momentum $k^m = (k^0, k^a, k^5) = (k, 0, 0, 0, 0, k)$, i.e. we have on the states $|k\,,\sigma\rangle$

$$P^0 = P^5 = k\,, \qquad P^a = 0\,, \quad a = 1, 2, 3, 4\,.$$

Further all operator formulas (written in the light-cone frame) should be understood as a result of their action on the subspace $\mathcal{H}_k \subset \mathcal{H}$ spanned by vectors $|k\,,\sigma\rangle$ with fixed light-cone momentum k_m.

The transition to this light-cone reference frame is conveniently performed in the light-cone basis where any 6D vector $X^m = (X^0, X^a, X^5)$ has the light-cone coordinates $X^m = (X^+, X^-, X^a)$, where

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^5\right), \quad X_{\pm} = \frac{1}{\sqrt{2}}(X_0 \pm X_5) \quad \Rightarrow \quad X^{\pm} = X_{\mp}, \tag{6}$$

and the contraction of two 6D vectors X^m and Y^m is

$$X^m Y_m = X^+ Y_+ + X^- Y_- + X^a Y_a = X_- Y_+ + X_+ Y_- - X_a Y_a . \tag{7}$$

In the light-cone basis the components of P have eigenvalues

$$P^+ = P_- = \sqrt{2}k, \qquad P^- = P_+ = 0, \qquad P^a = 0, \quad a = 1, 2, 3, 4 .$$

The higher Casimirs in this frame take the form

$$\begin{aligned} \hat{C}_4 &= -\hat{\Pi}_a \hat{\Pi}_a , \\ \hat{C}_6 &= \hat{\Pi}_b M_{ba} \hat{\Pi}_c M_{ca} - \tfrac{1}{2} M_{bc} M_{bc} \hat{\Pi}_a \hat{\Pi}_a , \end{aligned} \tag{8}$$

where we introduce Hermitian operators $\hat{\Pi}_a := \sqrt{2}kM_{+a}$ – vectors in 4D. Note that derivation of (8) takes some efforts.

The 4D operators $\hat{\Pi}_a$ and M_{ab} in view of (5) form the Lie algebra $\mathfrak{iso}(4)$

$$[\hat{\Pi}_a, \hat{\Pi}_b] = 0\,, \qquad [\hat{\Pi}_a, M_{bc}] = i\left(\delta_{ab}\hat{\Pi}_c - \delta_{ac}\hat{\Pi}_b\right)\,, \tag{9}$$

$$[M_{ab}, M_{cd}] = i\left(\delta_{bc}M_{ad} - \delta_{bd}M_{ac} + \delta_{ac}M_{db} - \delta_{ad}M_{cb}\right)\,, \tag{10}$$

and therefore generate the isometries of the four-dimensional Euclidean space. As a result, the operators $\hat{C}_4$ and $\hat{C}_6$ defined in (8) are the Casimirs of the $\mathfrak{iso}(4)$ algebra.

It is known that six generators M_{ab} of rotations in the space $\mathbb{R}^4$ are decomposed into the sum

$$M_{ab} = M_{ab}^{(+)} + M_{ab}^{(-)}\,, \tag{11}$$

of (anti)selfdual parts $M_{ab}^{(\pm)} := \frac{1}{2}\left(M_{ab} \pm \frac{1}{2}\,\epsilon_{abcd}M_{cd}\right)$. They form the algebra $\mathfrak{so}(4)$

$$[M_{ab}^{(\pm)}, M_{cd}^{(\pm)}] = i\left(\delta_{bc}M_{ad}^{(\pm)} - \delta_{bd}M_{ac}^{(\pm)} + \delta_{ac}M_{db}^{(\pm)} - \delta_{ad}M_{cb}^{(\pm)}\right),$$
$$[M_{ab}^{(+)}, M_{cd}^{(-)}] = 0\,,$$

which is direct sum $\mathfrak{su}(2) + \mathfrak{su}(2)$, where $M_{ab}^{(+)}$ and $M_{ab}^{(-)}$ generate the first and second algebras $\mathfrak{su}(2)$ respectively.

By using the 't Hooft symbols $\eta^{\mathrm{i}}_{ab} = -\eta^{\mathrm{i}}_{ba}$, and $\bar{\eta}^{\mathrm{i}'}_{ab} = -\bar{\eta}^{\mathrm{i}'}_{ba}$, $(\mathrm{i} = 1, 2, 3)$:

$$\eta^{\mathrm{i}}_{ab} = \begin{cases} \epsilon_{\mathrm{i}ab} & a, b = 1, 2, 3, \\ \delta_{\mathrm{i}a} & b = 4, \end{cases} \qquad \bar{\eta}^{\mathrm{i}'}_{ab} = \begin{cases} \epsilon_{\mathrm{i}'ab} & a, b = 1, 2, 3, \\ -\delta_{\mathrm{i}'a} & b = 4. \end{cases}$$

we connect (anti-)selfdual $SO(4)$ tensors $M^{(\pm)}_{ab}$ with the $SO(3)$ vectors $M^{(+)}_{\mathrm{i}}$, $M^{(-)}_{\mathrm{i}'}$ by means of the following relations

$$M^{(+)}_{ab} = -\eta^{\mathrm{i}}_{ab} M^{(+)}_{\mathrm{i}} , \qquad M^{(-)}_{ab} = -\bar{\eta}^{\mathrm{i}'}_{ab} M^{(-)}_{\mathrm{i}'} . \tag{12}$$

Operators $M^{(+)}_{\mathrm{i}}$ and $M^{(-)}_{\mathrm{i}'}$ form two $\mathfrak{su}(2)$ algebras with standard commutators

$$[M^{(+)}_{\mathrm{i}}, M^{(+)}_{\mathrm{j}}] = i\epsilon_{\mathrm{ijk}} M^{(+)}_{\mathrm{k}} , \quad [M^{(-)}_{\mathrm{i}'}, M^{(-)}_{\mathrm{j}'}] = i\epsilon_{\mathrm{i}'\mathrm{j}'\mathrm{k}'} M^{(-)}_{\mathrm{k}'} , \quad [M^{(+)}_{\mathrm{i}}, M^{(-)}_{\mathrm{j}'}] = 0 .$$

Recall that the algebra $\mathfrak{iso}(4)$ with basis elements Π_a and $M^{(\pm)}_{ab}$ generate stability subgroup for 6D massless unitary irreps.

In case of this noncompact symmetry there are two different cases which are defined by the value of the Casimir operator $C_4 = \hat{\Pi}_a \hat{\Pi}_a$, i.e. square of "four-translation" generator $\hat{\Pi}_a$ in $\mathbb{R}^4$. So, below we consider the following unitary massless representations.

- **Finite spin ("helicity") representations**.
  In these cases the $\mathrm{SO}(4)$ four-vector $\hat{\Pi}_a$ has zero square:

$$\hat{\Pi}_a \hat{\Pi}_a = 0\,. \tag{13}$$

- **Infinite (continuous) spin representations**.
  In case of these representations the Euclidean four-vector $\hat{\Pi}_a$ has nonzero square:

$$\hat{\Pi}_a \hat{\Pi}_a = \mu^2 \neq 0\,. \tag{14}$$

Below we consider these cases in details.

Massless finite spin representations for $\mathfrak{iso}(1,5)$ are characterized by the fulfillment of condition (13): $\hat{\Pi}_a\hat{\Pi}_a = 0$, which implies for Euclidean 4-vector:

$$\hat{\Pi}_a = 0 \quad , \qquad \forall\, a = 1,2,3,4\,. \tag{15}$$

As result, the Casimir operators $\hat{C}_4$ and $\hat{C}_6$ vanish in this case

$$\hat{C}_4 = -\hat{\Pi}_a\hat{\Pi}_a = 0\,, \qquad \hat{C}_6 = \hat{\Pi}_b M_{ba}\,\hat{\Pi}_c M_{ca} \;-\; \frac{1}{2} M_{bc}M_{bc}\,\hat{\Pi}_a\hat{\Pi}_a = 0\,,$$

In passing from this light-cone reference frame to an arbitrary basis, we get that all Casimir operators on the massless finite spin states take zero values [L.Mezincescu, A.Routh, P.Townsend, Annals Phys. **346** (2014) 66]

$$C_4 \equiv \frac{1}{24}W^2 = 0, \qquad C_6 \equiv \frac{1}{2^6}\Upsilon^k\Upsilon_k = 0\,.$$

Due to (15) the Euclidean four-translations are zero for these representations. As a result such unitary representations are finite dimensional (they are induced from irreps of the compact Lie algebra $\mathfrak{so}(4)$). Each such 6D massless representation defines the finite number of massless particle states.

6D helicity operators

Now we show that the Casimir operators of the stability group $SO(4)$ define the 6D helicity operators.
First, consider the vector $\Upsilon_m = \varepsilon_{mnklpr} P^n M^{kl} M^{pr}$. In the light-cone reference frame $P^+ = \sqrt{2}k\,, \qquad P^- = 0\,, \qquad P^a = 0$ the vector Υ_m has the components

$$\Upsilon^+ = \Lambda_1 P^+\,, \qquad \Upsilon^- = \Upsilon_a = 0\,, \tag{16}$$

where coefficient Λ_1 is the Casimir operator for $\mathfrak{so}(4)$

$$\Lambda_1 \;:=\; \epsilon_{abcd} M_{ab} M_{cd}\,. \tag{17}$$

We can write (16) to the general momentum frame. Namely, in view of $[\Upsilon_m, P_k] = 0$, $\Upsilon_k P^k = 0$ and since Υ_m is light like vector, we have:

$$\Upsilon_m = \Lambda_1 P_m\,, \quad \Lambda_1 := \frac{\Upsilon_0}{P_0}\,, \tag{18}$$

We stress that the operator Λ_1 is a central element in $\mathcal{U}(\mathfrak{iso}(1,5))$ since it is invariant under the 6D Poincare transformations.

Recall that $\mathfrak{so}(4)$ irreps are characterized by two second Casimir operators $(M^{(\pm)})^2$. The second operator appears as helicity operator Λ_2 in the construction proposed recently in [S.Kuzenko, A.Pindur, Phys.Lett. B 812 (2021) 136020; arXiv:2010.07124]. In their prescription the different third order vector is considered

$$S_m \; := \; 3M^{nk}P_{[m}M_{nk]} \; = \; M^{nk}M_{nk}P_m - 2M^{kn}M_{mn}P_k \, , \qquad (19)$$

where square brackets denote antisymmetrization. For this vector we obtain

$$\begin{aligned} S^m S_m &= M^2 M^2 P^2 + 4\Big[\Pi^k M_{km}\Pi_l M^{lm} - M^2(\Pi^2 + P^2) + \Pi^2\Big] \, , \\ P^m S_m &= M^2 P^2 - 2\Pi^2 \, , \quad [S_m, P_n] = 2iM_{mn}P^2 + 4i\Pi_{[m}P_{n]} \, . \end{aligned} \qquad (20)$$

where $M^2 := M^{nm}M_{nm}$ and $\Pi^2 := \Pi^l\Pi_l$. From these relations on the shell of the conditions $P^+ = \sqrt{2}k$, $P^- = P^a = 0$ and $\hat{\Pi}_a = 0$ $(\forall a)$, which define finite spin representations, we obtain

$$P^m S_m = 0 \, , \qquad [S_m, P_n] = 0 \, , \qquad S^m S_m = 0 \, , \qquad (21)$$

which are the same as conditions for Υ_n.

So, in case of massless finite spin representations, vectors P_m and S_m are collinear as well

$$S_m = \Lambda_2 P_m \,, \tag{22}$$

where the coefficient Λ_2 is the second $\mathfrak{so}(4)$ quadratic Casimir operator

$$\Lambda_2 \;:=\; M_{ab}M_{ab}\,, \qquad \Lambda_2 := \frac{S_0}{P_0}\,. \tag{23}$$

which is the second helicity operator. So these massless irreps are characterized by the pair (λ_1, λ_2), where $\lambda_{1,2} \in \mathbb{R}$ are eigenvalues of $\Lambda_{1,2}$. We can represent helicity operators $\Lambda_{1,2}$ in the form

$$\Lambda_1 = 2\left(M^{(+)}_{ab}M^{(+)}_{ab} - M^{(-)}_{ab}M^{(-)}_{ab}\right) = 8\left(M^{(+)}_{\mathrm{i}}M^{(+)}_{\mathrm{i}} - M^{(-)}_{\mathrm{i}'}M^{(-)}_{\mathrm{i}'}\right)\,,$$

$$\Lambda_2 = M^{(+)}_{ab}M^{(+)}_{ab} + M^{(-)}_{ab}M^{(-)}_{ab} = 4\left(M^{(+)}_{\mathrm{i}}M^{(+)}_{\mathrm{i}} + M^{(-)}_{\mathrm{i}'}M^{(-)}_{\mathrm{i}'}\right)\,.$$

For unitary irreps, the operators $M^{(+)}_{\mathrm{i}}M^{(+)}_{\mathrm{i}}$ and $M^{(-)}_{\mathrm{i}'}M^{(-)}_{\mathrm{i}'}$ are equal to $j_+(j_+ + 1)$ and $j_-(j_- + 1)$, respectively, and eigenvalues $\lambda_{1,2}$ are

$$\begin{aligned} \lambda_1 &= 8j_+(j_+ + 1) - 8j_-(j_- + 1)\,, \\ \lambda_2 &= 4j_+(j_+ + 1) + 4j_-(j_- + 1)\,, \qquad j_\pm \in \mathbb{Z}_{\geq 0}/2\,. \end{aligned} \tag{24}$$

Examples

First, we consider a fixed $\mathfrak{so}(4)$ irrep and determine the values of the helicities related to this irrep. We use the defining (vector) representation for the $\mathfrak{so}(4)$ generators:

$$(\mathcal{M}_{ab})_{eg} = i(\delta_{ae}\delta_{bg} - \delta_{ag}\delta_{be}) \,. \tag{25}$$

Then we reconstruct the corresponding $6D$ fields, for which the equations of motion and gauge fixing show that the independent components are exactly those $SO(4)$ fields which were considered earlier in the four-dimensional picture.

1. Vector electromagnetic field

For the representation (25) the $\mathfrak{so}(4)$ Casimir operators take the form

$$\begin{aligned} (\Lambda_1)_{eg} &= \epsilon_{abcd}(M_{ab}M_{cd})_{eg} = 0 \,, \\ (\Lambda_2)_{eg} &= (M_{ab}M_{ab})_{eg} = 6\delta_{eg} \,. \end{aligned} \tag{26}$$

and their action on the 4-dimensional vector field A_a gives the following values of helicities: $\lambda_1 = 0 \,, \quad \lambda_2 = 6 \,; \qquad j_+ = j_- = \frac{1}{2}$.

This Euclidean 4D vector field A_a describes physical components of the 6D vector gauge field A_m. Indeed, in momentum representation $U(1)$ gauge field A_m is determined up to gauge transformations

$$\delta A_m = iP_m \varphi \tag{27}$$

and described by the equations of motion

$$P^m F_{mn} = 0\,, \tag{28}$$

where $F_{mn} = i(P_m A_n - P_n A_m)$ is the field strength. The proper gauge fixing is the light-cone gauge

$$A^+ = 0\,. \tag{29}$$

Then in the light-cone frame

$$P^+ = \sqrt{2}k\,,\; P^- = P^a = 0\,,$$

the equations of motion 28 give $A^- = 0$ and independent field is the transverse part A_a of the 6D gauge field A_m.

2. Linearized gravity field in 6D

Now we consider the space of $SO(4)$ second rank tensors. In this case the matrix representation of $so(4)$ generators is

$$(M_{ab})_{e_1e_2,g_1g_2} = (\mathcal{M}_{ab})_{e_1g_1}\delta_{e_2g_2} + \delta_{e_1g_1}(\mathcal{M}_{ab})_{e_2g_2} \tag{30}$$

and the $SO(4)$ Casimir operators are

$$\begin{aligned} (\Lambda_1)_{e_1e_2,g_1g_2} &= \epsilon_{abcd}(M_{ab}M_{cd})_{e_1e_2,g_1g_2} = 8\epsilon_{e_1e_2g_1g_2}\,, \\ (\Lambda_2)_{e_1e_2,g_1g_2} &= (M_{ab}M_{ab})_{e_1e_2,g_1g_2} \\ &= 4(3\delta_{e_1g_1}\delta_{e_2g_2} + \delta_{e_1g_2}\delta_{e_2g_1} - \delta_{e_1e_2}\delta_{g_1g_2})\,. \end{aligned} \tag{31}$$

First, we consider the $SO(4)$ second rank tensor $\hat{h}_{ab}$, which is symmetric $\hat{h}_{ab} = \hat{h}_{ab}$ and traceless $\hat{h}_{aa}$. On this field the helicity operators (31) take the values

$$\lambda_1 = 0\,, \quad \lambda_2 = 16\,; \qquad j_+ = j_- = 1\,. \tag{32}$$

Let us show that this field $\hat{h}_{ab}$ describes the physical components of the $6D$ linearized gravitational field.

Let us show that this 4D field $\hat{h}_{ab}$ describes the physical components of the 6D linearized gravitational field. Indeed, the 6D linearized gravitational field $h^{mn} = h^{nm}$ has gauge invariance

$$\delta h^{mn} = iP^{(m}\varphi^{n)} \tag{33}$$

and obeys the Pauli-Fierz equations of motion

$$P^2 h^{mn} - P^m P_k h^{nk} - P^n P_k h^{mk} + P^m P^n h_k{}^k = 0\,. \tag{34}$$

For the transformations (33) we can put again the light-cone gauge (see e.g. [W. Siegel, *Fields*, hep-th/9912205])

$$h^{+m} = 0\,. \tag{35}$$

The equations of motion (34) give $h^{-m} = 0$, $h_a{}^a = 0$ in the light-cone frame. As result, nonvanishing field is the traceless part $\hat{h}_{ab}$ of transverse field h_{ab} of the 6D gravity field h^{mn}.

3. Third rank (anti-)selfdual antisymmetric tensor fields

Consider the $SO(4)$ antisymmetric tensors of the second rank $B_{ab} = -B_{ba}$, and their (anti-)selfdual parts

$$B^{(\pm)}_{ab} = \pm \frac{1}{2}\, \epsilon_{abcd} B^{(\pm)}_{cd} \, . \tag{36}$$

The tensors $B^{(\pm)}_{ab}$ form the spaces of two $SO(4)$ irreps which make up the $SO(4)$ reducible representation in the space of all antisymmetric rank 2 tensors associated to Young diagram $[1^2] \equiv \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$. In this case the $\mathfrak{so}(4)$ generators M_{ab} and helicity operators Λ_1, Λ_2 have the same representations as in the previous case. Eigenvalues of Λ_1, Λ_2 and $(M^{(\pm)}_{\rm i})^2$ are

$$\begin{array}{ll} \lambda_1 = 16\, , \quad \lambda_2 = 8\, ; \qquad j_+ = 1\, , \quad j_- = 0\, , \\ \lambda_1 = -16\, , \quad \lambda_2 = 8\, ; \qquad j_+ = 0\, , \quad j_- = 1 \end{array} \tag{37}$$

on the spaces of the selfdual $B^{(+)}_{ab}$ anti-selfdual $B^{(-)}_{ab}$ fields.

It is clear that these $SO(4)$ (anti-)selfdual fields $B^{(\pm)}_{[ab]}$ are independent components of the $6D$ massless (anti-)selfdual 3-rank fields $B^{(\pm)}_{mnk}$ which satisfy the identities

$$B^{(\pm)}_{mnk} = \pm\frac{1}{3!}\,\varepsilon_{mnklpr}B^{(\pm)lpr}\,. \qquad (38)$$

Indeed, the equations of motion of the $6D$ massless fields $B^{(\pm)}_{mnk}$ are

$$\text{a)}\;\; P^m B^{(\pm)}_{mnk} = 0\,, \qquad \text{b)}\;\; P_{[m}B^{(\pm)}_{nkl]} = 0\,, \qquad \text{c)}\;\; P^2 B^{(\pm)}_{nkl} = 0\,. \qquad (39)$$

Then in the light-cone frame the equations (39a) give $B^{(\pm)}{}_{+mn} = 0$ whereas the equations (39b) produce $B^{(\pm)}{}_{abc} = 0$. As a result, independent fields of the $6D$ tensors $B^{(\pm)}_{mnk}$ are the $SO(4)$ (anti-)selfdual fields $B^{(\pm)}{}_{-ab} \equiv B^{(\pm)}{}_{ab}$ which are subjected the $SO(4)$ (anti-)selfdual conditions (36) due to the $6D$ (anti-)selfdual conditions (38).

Remark. One can generalize this example to the case of special $3n$-rank selfdual and anti-selfdual 6-dimensional tensor fields. These fields correspond to $SO(4)$ irreducible representations in spaces of $2n$-rank traceless selfdual and anti-selfdual tensors with components $B^{(\pm)}_{a_1 \dots a_{2n}}$ be symmetrized in accordance to the Young diagram $[n^2] \equiv \begin{array}{|c|c|c|c|} \hline & & \cdots & \\ \hline & & \cdots & \\ \hline \end{array}$. It is clear that for highest weights of such selfdual and anti-selfdual representations of $SO(4)$ we have respectively $j_+ = n, j_- = 0$ and $j_+ = 0, j_- = n$ and in view of general formulas (24) we obtain the eigenvalues of helicity operators

$$\lambda_1 = 8n(n+1), \;\; \lambda_2 = 4n(n+1) \; ,$$
$$\lambda_1 = -8n(n+1), \;\; \lambda_2 = 4n(n+1) \; ,$$

which generalize (37).

Massless infinite (continuous) spin irreps

Here we have the condition (14)

$$\hat{\Pi}_a \hat{\Pi}_a = \mu^2 \neq 0\,, \tag{40}$$

and the Euclidean four-vector $\hat{\Pi}_a$ is nonzero. Then the representations of the little group $ISO(4)$, from which the $6D$ relativistic massless representations are induced, are infinite dimensional. In these representations the operator C_4 has nonvanishing eigenvalue

$$C_4 \; = \; \hat{C}_4 \; = \; -\mu^2\,, \qquad \mu \neq 0\,. \tag{41}$$

Moreover, for the condition (40) we can take the basis in which only the fourth component is nonzero: $\hat{\Pi}_1 = \hat{\Pi}_2 = \hat{\Pi}_3 = 0, \quad \hat{\Pi}_4 = \mu$. Then taking into account $\eta^{\rm i}_{a4} = \delta_{{\rm i}a}$ and $\bar{\eta}^{\rm i'}_{a4} = -\delta_{{\rm i'}a}$ we obtain the value of the Casimir operator:

$$\hat{C}_6 \; = \; -\mu^2 J_{\rm i} J_{\rm i}\,, \tag{42}$$

where

$$J_{\rm i} \; := \; M^{(+)}_{\rm i} + M^{(-)}_{\rm i} \; = \; -\frac{1}{2}\,\epsilon_{\rm ijk} M_{\rm jk}\,, \qquad {\rm i} = 1,2,3\,. \tag{43}$$

So the operators J_i are in fact the generators of the $\mathrm{SO}(3)$ subgroup of the $\mathrm{SO}(4)$ stability group. Therefore, in case of the unitary irreps we have

$$J^2 \; = \; s(s+1)\,, \tag{44}$$

where s is fixed integer or half-integer number. Thus, for the Casimir operator (42) we obtain

$$C_6 \; = \; \hat{C}_6 \; = \; -\mu^2\, s(s+1)\,, \tag{45}$$

As a result, the massless infinite spin representations are characterized by the pair (μ, s), where the real parameter μ defines the eigenvalue of the Casimir operator C_4 (41) and the (half-)integer number s defines the eigenvalue of the Casimir operator C_6 (45).
Let us examine in our consideration the $D=6$ infinite integer spin system [X.Bekaert, J.Mourad, JHEP **0601** (2006)115, hep-th/0509092]. which is higher dimension generalization of the $D=4$ Wigner - Bargman model [E.Wigner (1939,1947), V.Bargmann and E.Wigner (1948)].

The Bekaert-Mourad model is described by the pair of the space-time phase operators

$$x^m\,, p_m\,, \quad [x^m, p_k] = i\delta^m_k \tag{46}$$

and two pairs of the additional bosonic phase vectors

$$w^m\,, \xi_m\,, \quad [w^m, \xi_k] = i\delta^m_k\,; \qquad u^m\,, \zeta_m\,, \quad [u^m, \zeta_k] = i\delta^m_k\,. \tag{47}$$

These two pairs of vectors (47) are responsible for spinning degrees of freedom.

Infinite integer spin field Ψ in Bekaert-Mourad model is described by the $D = 6$ generalization of the Wigner-Bargmann equations

$$p^2\,\Psi = 0\,, \qquad \xi{\cdot}p\,\Psi = 0\,, \tag{48}$$

$$(w{\cdot}p - \mu)\,\Psi = 0\,, \qquad (\xi{\cdot}\xi + 1)\,\Psi = 0\,, \tag{49}$$

and additional equations which involve the second pair (47)

$$u{\cdot}p\,\Psi = 0\,, \qquad \zeta{\cdot}p\,\Psi = 0\,, \tag{50}$$

$$\zeta{\cdot}\xi\,\Psi = 0\,, \qquad \zeta{\cdot}\zeta\,\Psi = 0\,, \qquad (u{\cdot}\zeta - s)\,\Psi = 0\,, \tag{51}$$

where $\xi{\cdot}p := \xi^m p_m$, *etc*.

Note that, in contrast to the four-dimensional Wigner-Bargman model with one pair of auxiliary variables w^m, ξ_m, in the six-dimensional case it is necessary to use the second pair of auxiliary vector variables u^m, ζ_m to describe arbitrary infinite spin representations.
In the light-cone frame $p^- = p_a = 0$, $p^+ = \text{const} \neq 0$, and in the representation $\xi_m = -i\partial/\partial w^m$, $\zeta_m = -i\partial/\partial u^m$ the equations (48)–(51) can be solved as

$$\Psi = \delta(p^+w^- - \mu)\,\delta(p^+u^-)\,\Phi(w_a, u_a)\,, \tag{52}$$

where $\Phi(w_a, u_a)$ has special series expansions presented in [X.Bekaert, J.Mourad, JHEP **0601** (2006)115, hep-th/0509092].
Now we can determine the values of the Casimir operators C_4, C_6 on the field (52). For such fields the generators of the $iso(4)$ algebra have the form

$$M_{ab} = i\left(w_a\,\frac{\partial}{\partial w_b} - w_b\,\frac{\partial}{\partial w_a} + u_a\,\frac{\partial}{\partial u_b} - u_b\,\frac{\partial}{\partial u_a}\right)\,, \qquad \hat{\Pi}_a = -i\mu\frac{\partial}{\partial w_a}\,. \tag{53}$$

As result, we obtain the fulfillment of the condition (41) for the Casimir operator C_4: $C_4 = \hat{C}_4 = -\mu^2$. Moreover, the representations (53) lead to the expression

$$\begin{aligned} \hat{C}_6 &= \mu^2 u_a \frac{\partial}{\partial u_a}\left(u_b \frac{\partial}{\partial u_b} + 1\right)\frac{\partial}{\partial w_c}\frac{\partial}{\partial w_c} \\ &+ \mu^2 \left(u_a \frac{\partial}{\partial w_a} u_b \frac{\partial}{\partial w_b} - u_a u_a \frac{\partial}{\partial w_b}\frac{\partial}{\partial w_b}\right)\frac{\partial}{\partial u_c}\frac{\partial}{\partial u_c} \\ &+ \mu^2 \left(u_a u_a \frac{\partial}{\partial u_b}\frac{\partial}{\partial w_b} - 2u_a \frac{\partial}{\partial u_a} u_b \frac{\partial}{\partial w_b}\right)\frac{\partial}{\partial u_c}\frac{\partial}{\partial w_c} \end{aligned} \tag{54}$$

and due to the equations (48)–(51) we obtain $C_6 = \hat{C}_6 = -\mu^2 s(s+1)$ on the fields (52).

Thus, the infinite spin field with only one additional vector variables and obeying the Wigner-Bargmann equations (48)–(49) and additional equations (50)–(51) describes the irreducible (μ, s) infinite spin representation.

Summary and outlook

We have studied the massless irreducible representations of the Poincaré group in six-dimensional Minkowski space and give full classification of all massless representations including infinite integer spin case. The representations are described by three Casimir operators C_2, C_4 and C_6. The different forms and properties of these operators are explored in the standard massless momentum reference frame, where it is seen that the unitary representations of $ISO(1,5)$ group are induced from representations of $SO(4)$ and $ISO(4)$ groups and correspondingly are divided into finite spin (helicity) and infinite spin representations. Both these representations are studied in details. It is proved that the finite spin representation is described by two integer or half-integer numbers while the infinite spin representation is described by one real parameter and one integer or half-integer number. In case of half-integer spin we should introduce an additional spinor or twistor variables like in [X.Bekaert, J.Mourad, JHEP **0601** (2006)115, hep-th/0509092].

As a continuation of this research it would be interesting to describe the massless representations with half-integer spin and massive irreducible representations of six-dimensional Poincaré group with both integer and half-integer spin. Another open problem is constructing the representations of the corresponding six-dimensional *super* Poincaré group. Also it would be useful to work out the field realizations of the massless representations considered in this paper and explore the new aspects of Lagrange formulation for these fields in six-dimensional Minkowski space including infinite spin cases. We plan to study all these problems in the forthcoming papers.