TT-deformed 2d YM theory at large N: Collective field theory and phase transitions

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The Plan of the talk

- 2d YM theory,generalities. Partition function, fermionic representation,DK phase transition
- TT-deformation of 2d CFT
- TT-deformation of 2d YM. Collective field theory on the cylinder
- Solutions to EOM for collective field theory on the sphere.1-st order phase transition
- DK phase transition on the disc in TT deformed YM
- DK transition in TT deformed q-YM

2D YM. Partition function

$$Z_g(g_{YM}, A) = \int [d\mathbf{A}]\,[d\boldsymbol{\Phi}] \exp\left(\int_{\Sigma_g} d\sigma dt \left(\mathrm{Tr}\,\boldsymbol{\Phi}\mathbf{F} + \frac{g_{YM}^2}{N}\mathrm{Tr}\,\boldsymbol{\Phi}^2\right)\right).$$

$$Z_g(g_{YM}, A) = \sum_R (\dim R)^{2-2g} e^{-\frac{A g_{YM}^2}{2N} C_2(R)},$$

$\overline{R}$

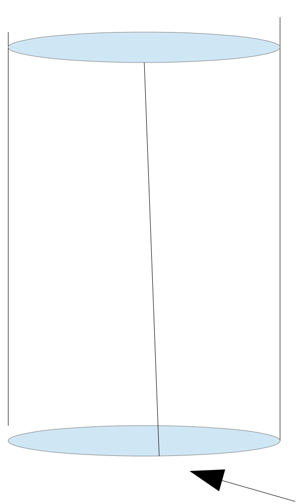
(Migdal 75)

$$C_2(R) = \sum_{i=1}^{N} n_i\,(n_i - 2i + N + 1)\,, \qquad \dim R = \prod_{i>j}\left(1 - \frac{n_i - n_j}{i - j}\right).$$

Partition function for a cylinder

$$Z_{cyl}(g^2_{YM}, A|U_1, U_2) = \sum_R \chi_R(U_1)\chi_R(U_2^\dagger)\, e^{-\frac{Ag^2_{YM}}{2N}C_2(R)+i\theta|R|},$$

$$\chi_R(U) = \frac{\det_{i,j}(e^{i(n_i-i)\theta_j})}{\prod_{i<j}(e^{i\theta_i} - e^{i\theta_j})},$$



2d YM on cylinder with Wilson line in one-row representation= trigonometric Calogero model

(Nekrasov, .G. 93)

Heavy Wilson line

Collective field theory

$$\frac{\partial S}{\partial A} = \frac{1}{2}\int_0^{2\pi} \sigma_1(\theta)\left[\left(\frac{\partial}{\partial\theta}\frac{\delta S}{\delta\sigma_1(\theta)}\right)^2 - \frac{\pi^2}{3}\sigma_1^2(\theta)\right].$$

$$H\left[\sigma(\theta), \Pi(\theta)\right] = \frac{1}{2}\int_0^{2\pi} \sigma(\theta)\left[\left(\frac{\partial\Pi}{\partial\theta}\right)^2 - \frac{\pi^2}{3}\sigma^2(\theta)\right]$$

Das-Jevicki Hamiltonian

$$\frac{\partial\sigma(\theta)}{\partial t} = \frac{\delta H[\sigma,\Pi]}{\delta\Pi(\theta)}, \qquad \frac{\partial\Pi(\theta)}{\partial t} = -\frac{\delta H[\sigma,\Pi]}{\delta\sigma(\theta)}$$

2D YM. Collective field theory

$$\begin{cases} \dfrac{\partial\sigma}{\partial t} + \dfrac{\partial}{\partial\theta}(\sigma v) = 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial\theta} = \dfrac{\partial}{\partial\theta}\left(\dfrac{\pi^2}{2}\sigma^2\right) \end{cases}, \quad v(\theta) \equiv \frac{\partial\Pi}{\partial\theta}.$$

$$f(t,\theta) = v(t,\theta) + i\pi\rho(t,\theta).$$

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial\theta} = 0$$

Equation in the 2d YM collective field theory (Gross,Matytsin 94)

2D YM. Douglas-Kazakov phase transition

$$h(x) = -\frac{1}{2} + \frac{i - n_i}{N}, \qquad x = \frac{i}{N}$$

$$S_{eff}[h(x)] = -\int_0^1 \int_0^1 dx dx' \log|h(x) - h(x')| + \frac{Ag_{YM}^2}{2} \int_0^1 dx\, h^2(x) - \frac{Ag_{YM}^2}{24}.$$

$$\rho(h) \leq 1 \qquad\qquad \rho(h) = \frac{\partial x(h)}{\partial h}.$$

$$\frac{\delta S_{\text{eff}}[h(x)]}{\delta h(x)} = 0 \quad \Rightarrow \quad \rho = \frac{Ag_{YM}^2}{2\pi} \sqrt{\frac{4}{Ag_{YM}^2} - h^2}.$$

Douglas-Kazakov,93

DK phase transition

$$A \equiv A_{cr} = \frac{\pi^2}{g_{YM}^2}.$$

3-rd order transition

$$\Delta F \sim \frac{g_{YM}^6}{2\pi^6}(A - A_{cr})^3$$

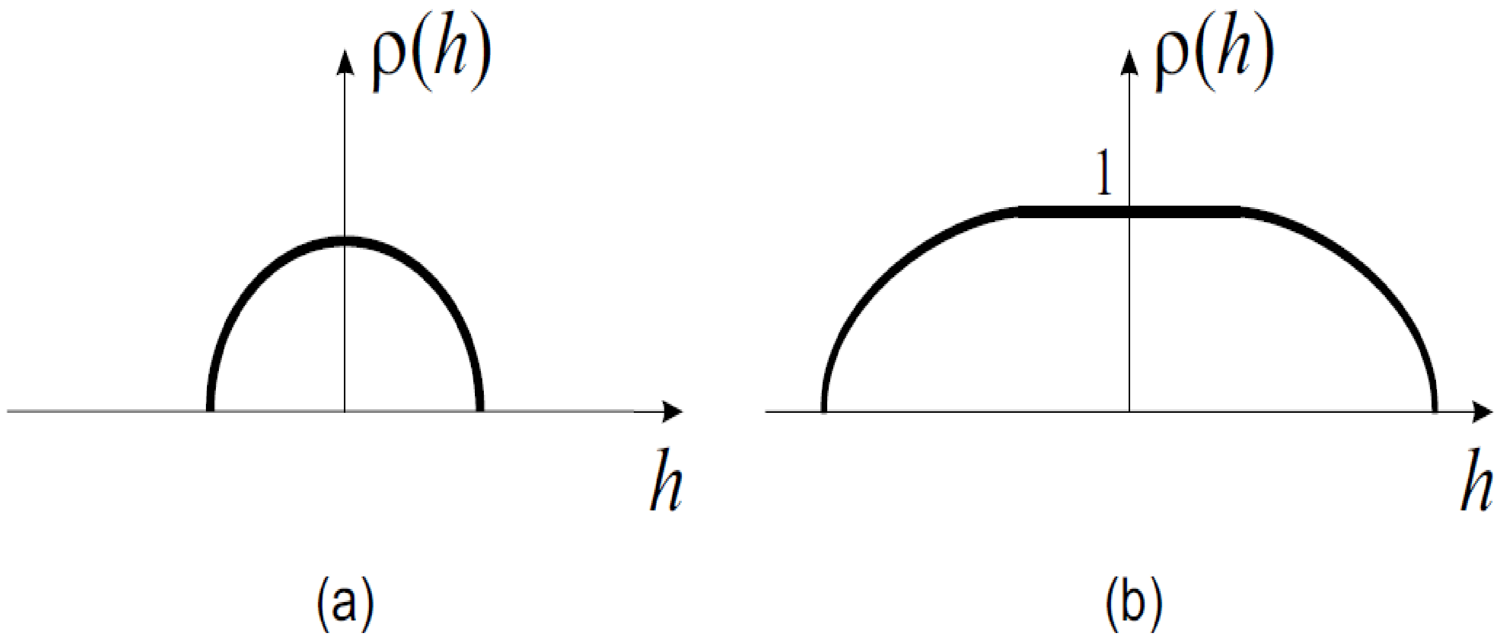


Figure 1: Density profile: (a) below the phase transition, (b) above the phase transition.

2D YM. Douglas-Kazakov phase transitions. Instantons

$$Z_g(g_{YM}, A, \theta) = \sum_{n} \Delta^{2-2g}(n) \exp\left(-\frac{A g_{YM}^2}{2N} \sum n_i^2 - i\theta \sum n_i\right)$$

Near the critical time, $t = t_c$, there are finite probabilities to get $M = 0, M = 1, M = -1$

$$\mathbb{P}(M=0) = 1 - \frac{q(s)}{2^{1/3} N^{1/3}} + \ldots; \qquad \mathbb{P}(M=1) = \mathbb{P}(M=-1) = \frac{q(s)}{2^{4/3} N^{1/3}} + \ldots$$

where $q(s)$ is the solution to the Painleve II equation

$$q''(s) = sq(s) + 2q(s)^3$$

Gross,Matytsin 94

At $t > t_c$, the probability to have a specific total winding number M is given by a Gaussian distribution centered at zero,

$$\mathbb{P}(M) = Ce^{-\kappa M^2} + \mathcal{O}(N^{-1}). \tag{66}$$

where C - is a mere normalization constans and κ is an implicit function of t, given in terms of full elliptic integrals:

$$\kappa = -\frac{\pi K(k')}{K(k)}, \qquad t = 8E(k)K(k) - (1-k^2)K^2(k), \qquad k' = \sqrt{1-k^2}. \tag{67}$$

2D YM. Fermionic representation.Vicious walkers

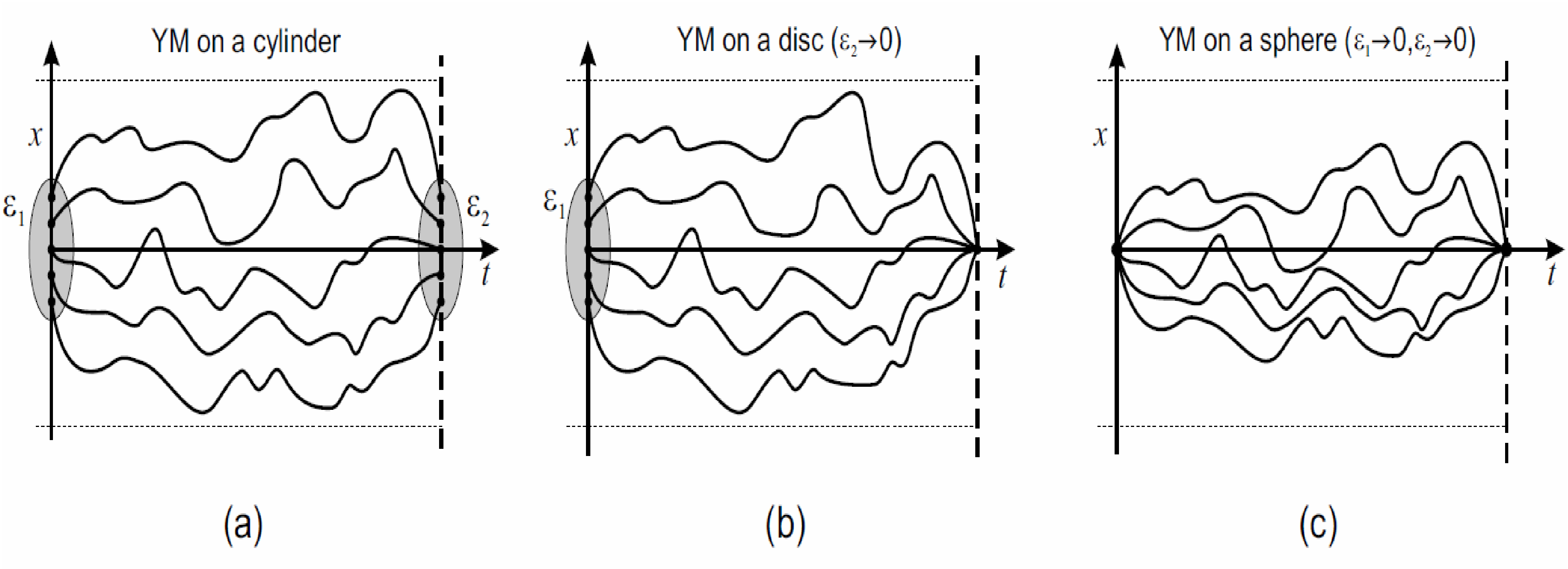


Figure 2: The 2D YM on a cylinder (a) is reduced to the 2D YM a disc (b) when left extremities are kept on finite support ($\varepsilon_2 \to 0$), and right ones are set to zero, and to the 2D YM on a sphere (c) when $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

(Milekhin,Nechaev, A.G. 19)

Vicious walkers

$$\Psi_N(\mathrm{p}|\mathrm{x}) = \frac{1}{\sqrt{N!}} \det \psi(p_j|x_k), \quad \psi(p_j|x_k) = e^{ip_j x_k} \qquad 1 \le j,k \le N$$

$$P_N(t,|\mathbf{x},\mathbf{y}=\mathbf{x})\Big|_{\mathbf{x}\to 0} = \langle \Psi_N | e^{-tH_0} | \Psi_N \rangle \Big|_{\mathbf{x}\to 0} = \sum_{\mathbf{p}} \Psi(\mathbf{p}|\mathbf{x})\, \Psi^*(\mathbf{p}|\mathbf{x})\, e^{-tE(\mathbf{p})}\Big|_{\mathbf{x}\to 0},$$

$$P_N(t,L|\mathbf{x}\to 0) = C\delta^{N(N-1)/2} \sum_{\mathbf{p}\in Z} \prod_{i<j}^{N} (p_i - p_j)^2 e^{-\frac{4\pi^2 Dt}{L^2} \sum_{i=1}^{N} p_i^2},$$

Solution to Fokker-Planck eq

Mapping of parameters (Forrester,Majumdar,Scher 13)

$$g_{YM}^2 A = \frac{8\pi^2 DNt}{L^2}$$

DK transition. Strong coupling phase — wrappings dominate

TT-deformation of CFT.Generalities

- TT-deformation is the controllable deformation of CFT by the irrelevant operator (Zamolodchikov,Smirnov)
- It corresponds to introduction of radial cut-off in the holographic picture(Gross et,al, Ilesiu et al)
- S-matrix in the deformed theory can be evaluated (Dubovsky,Gorbenko,Hernandez)
- The deformation applied to the point-like particles brings to the hard rods(Cardy et al)

TT-deformation of 2D YM

$$\partial_\tau \mathcal{L}(\tau) = -detT_{\mu\nu}(\tau), \quad T\bar{T}(\tau) = -\pi^2 detT_{\mu\nu}(\tau)$$

$$\mathcal{H}_\tau = \frac{\mathcal{H}_0}{1-\tau\mathcal{H}_0}$$

$$\hat{H}_0 = \frac{g_{YM}^2}{2}\int_0^L \frac{\delta^2}{(\delta A_1^a)^2}dx$$

Deformed 2dYM partition function (Dorey,Negroo,Tateo 18)

$$Z_N(U_1, U_2|A,\tau) = \sum_R \chi_R(U_1)\chi_R(U_2^\dagger)\exp\left(-\frac{\frac{Ag_{YM}^2}{2}C_2(R)}{1-\tau\frac{g_{YM}^2}{2}C_2(R)}\right)$$

Deformation of Das-Jevicki Hamiltonian and 1-st order phase transition

Large N thermodynamical limit

$$H[\sigma_1(\theta),\Pi(\theta)] = \frac{1}{2}\int_0^{2\pi} \sigma_1(\theta)\left[\left(\frac{\partial \Pi}{\partial \theta}\right)^2 - \frac{\pi^2}{3}\sigma_1^2(\theta)\right] d\theta, \quad \Pi(\theta) = \frac{\delta S}{\delta \sigma_1(\theta)}.$$

$$\widetilde{H} = \frac{H + \frac{1}{24}}{1 + 2\tau(H + \frac{1}{24})}$$

$$\frac{\partial S}{\partial A} + \widetilde{H} = 0$$

Deformed Das-Jevicky Hamiltonian

$$\begin{cases} \dfrac{\partial \sigma(\theta)}{\partial t} = \dfrac{\delta \widetilde{H}(\sigma,\Pi)}{\delta \Pi(\theta)} = \alpha \dfrac{\delta H(\sigma,\Pi)}{\delta \Pi(\theta)}, \\ \dfrac{\partial \Pi(\theta)}{\partial t} = -\dfrac{\delta \widetilde{H}(\sigma,\Pi)}{\delta \sigma(\theta)} = -\alpha \dfrac{\delta H(\sigma,\Pi)}{\delta \sigma(\theta)} \end{cases}$$

$$\alpha = \frac{\partial \widetilde{H}}{\partial H} = \frac{1}{(1 + 2\tau(H + \frac{1}{24}))^2}$$

Deformation of Das-Jevicki Hamiltonian and 1-st order phase transition

$$\sigma_1(\theta) = \sigma_2(\theta) = \delta(\theta). \qquad \sigma_*(t,\theta) = \frac{2}{\pi r^2(t)}\sqrt{r^2(t) - \theta^2} \quad |\theta| < r.$$

Sphere boundary condition

$$r(t) = 2\sqrt{\frac{t(A-t)}{A}b^{\pm}}, \quad \text{where} \quad b^{\pm}(A,\tau) = \frac{1 + \frac{2\tau}{A}(1+\frac{\tau}{12}) \pm \sqrt{1 + \frac{4\tau}{A}(1+\frac{\tau}{12})}}{2(1+\frac{\tau}{12})^2}.$$

Solutions to the EOM in collective fieeld theory

$$b^+ \to 1 \text{ and } b^- \to 0 \text{ when } \tau \to 0.$$

$$E_\lambda^{\pm}(R,A) = \frac{R/24}{1-\lambda/24} + \frac{R}{2\widetilde{\lambda}}\left(-1 \pm \sqrt{1 - \frac{4\widetilde{\lambda}}{2AR^2}}\right), \quad \widetilde{H}^{\pm} = E_\lambda^{\pm}(1,A).$$

Two branches

To keep unitarity of the theory it is necessary to take into account both
Branches of solutions for deformed theory

$$\partial_\lambda E^{\pm}_{\lambda}(R,A) = E^{\pm}_{\lambda}(R,A)\partial_R E^{\pm}_{\lambda}(R,A)$$

$$Z = e^{-\mathcal{F}^{+}} + e^{-\mathcal{F}^{-}}$$

Somewhat similar siltuation
With two solutions for JT gravity
At finite cut-off
(Ilesiu,Turiaci,Krutoff,Verlinde 19)

$$\mathcal{F}^{\pm} = -N^2\left(1 - \frac{1}{2}\log(Ab^{\pm}) + \frac{A}{2}\frac{\left(\frac{-1}{2Ab^{\pm}} + \frac{1}{24}\right) + \frac{1}{24}}{1 + 2\tau\left(\frac{-1}{2Ab^{\pm}} + \frac{1}{24}\right)}\right).$$

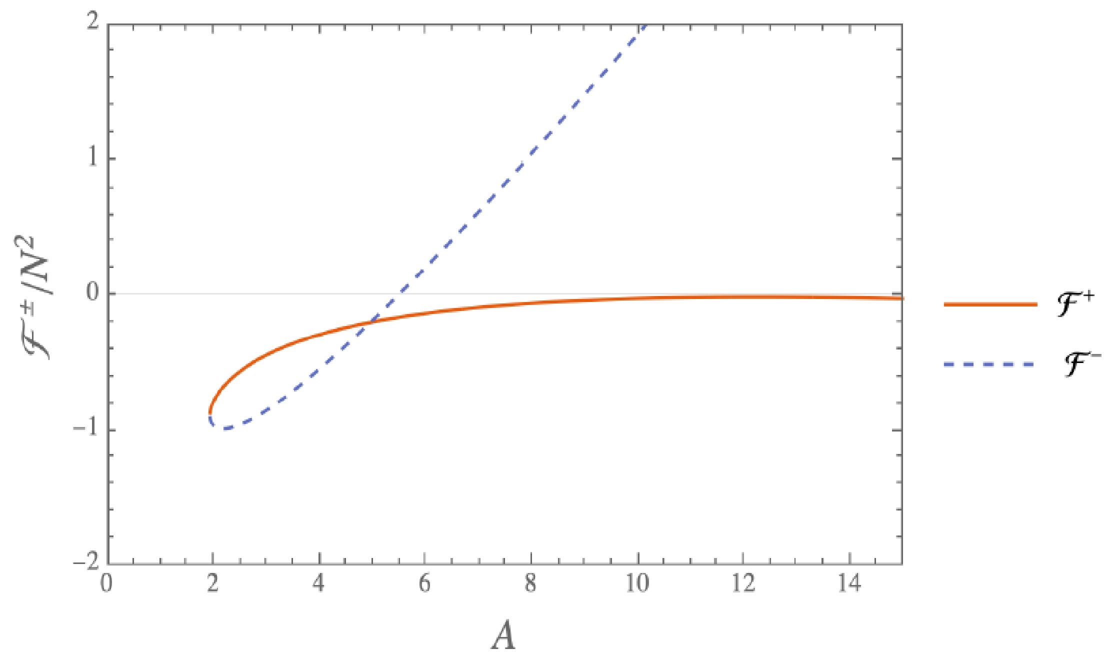


Figure 3: Two branches for $\tau = -0.5$ in the weak coupling phase. The smallest free energy dominates in the partition function. In the large N limit the first derivative with respect to A of the full partition function Z is discontinuous at $A_* \approx 4.948$.

DK phase transition in deformed theory on the disc

$$Z_N(U|A,\tau) = \sum_R \dim(R)\chi_R(U)e^{-\frac{A}{2N}\frac{C_2(R)}{1-\tau C_2(R)/N^3}}$$

$$A_{cr}^{(\infty)} = \pi A_{cr}^{(0)}\left[1-\tau\left(\int dh\rho_s^{(0)}(A_{cr}^{(0)},h)h^2-\frac{1}{12}\right)\right]^2$$

Critical area for DK transition on a disc
Transition in deformed YM as a function of holonomy

$$A_{cr}^{(0)} = \pi\left(\int \frac{\sigma(\theta)}{\pi-\theta}d\theta\right)^{-1}.$$

$$\begin{cases} G_+(G_-(x)) = G_-(G_+(x)) = x, \\ \mathrm{Im}\, G_+(h+i0) = \pi\rho_s^{(0)}(A,h), \\ \mathrm{Im}\, G_-(\theta+i0) = -\pi\sigma(\theta). \end{cases}$$

Critical area for the sphere — (Santilli,Tierz 19)

1-st order versus 3-rd order

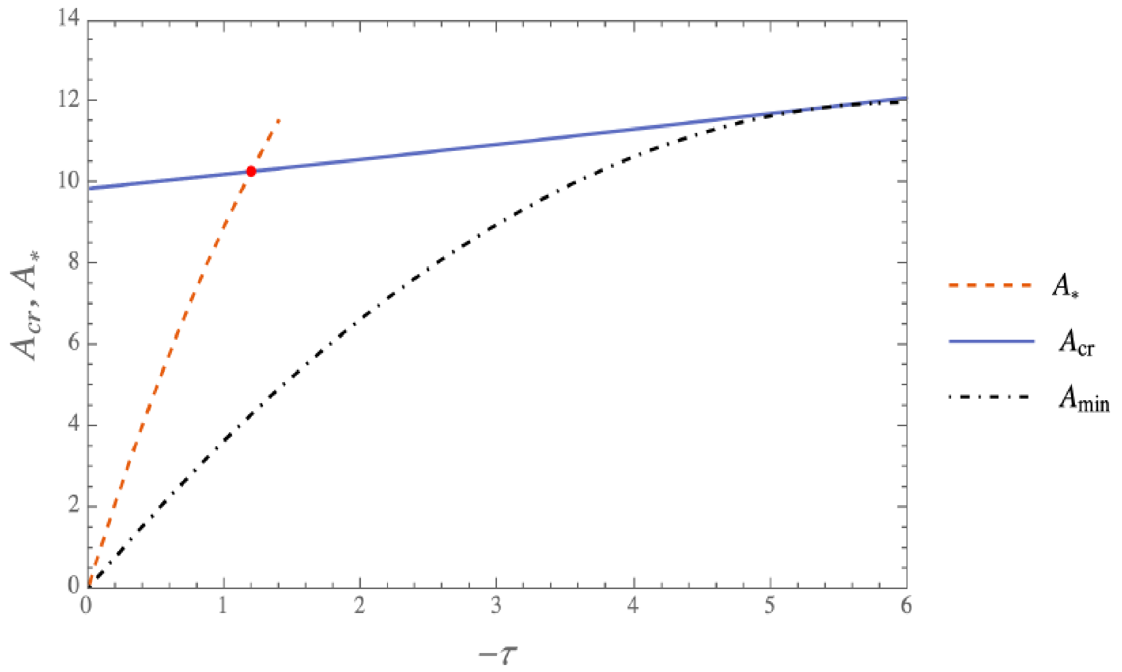


Figure 4: The positions of the Douglas-Kazakov and the 1st order phase transitions as functions of τ. The intersection point is $\tau_* \approx -1.196$. $A_{\rm min}$ corresponding to the shock singularity is shown.

DK phase transitions in TT-deformed q-YM theory

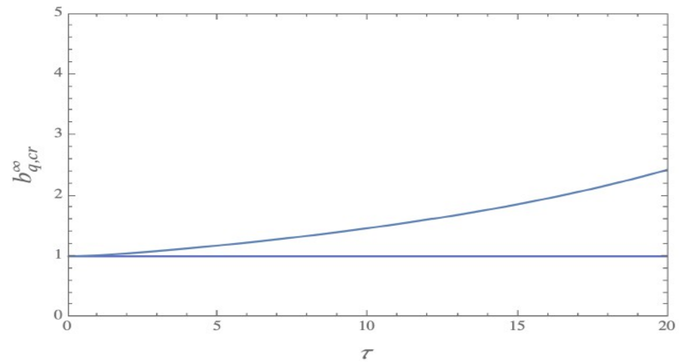
$$Z_{q\text{-}T\bar{T}} = \sum (\dim_q R)^2 q^{\frac{p}{2}\frac{C_2(R)}{1-\tau C_2(R)/N^3}}$$

$$S_{eff}[\rho] = -\int dh \int dv \rho(h)\rho(v) \log\left[2\sinh\frac{A(h-v)}{2p}\right] + \frac{A}{2}\frac{\int dh\rho(h)h^2 - \frac{1}{12}}{1-\tau\left(\int dh\rho(h)h^2 - \frac{1}{12}\right)} + \frac{2p^2}{A^2}F_0^{CS}(A/p)$$

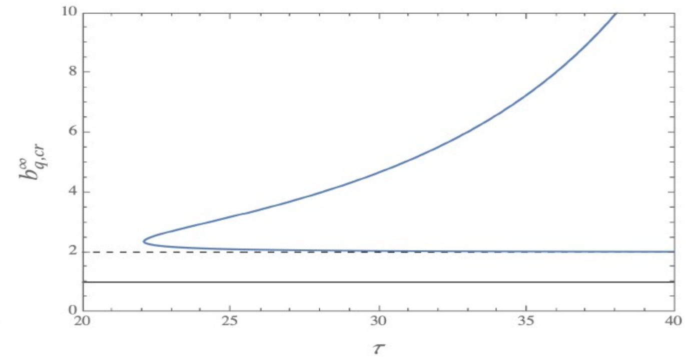
$$F_0^{CS}(t) = \frac{t^3}{12} - \frac{\pi^2 t}{6} - \mathrm{Li}_3\left(e^{-t}\right) + \zeta(3)$$

$$A_{cr}^{(\infty)} = -\left(pb_{q,cr}^{(\infty)}\right)^2 \log\cos^2\frac{\pi}{pb_{q,cr}^{(\infty)}}$$

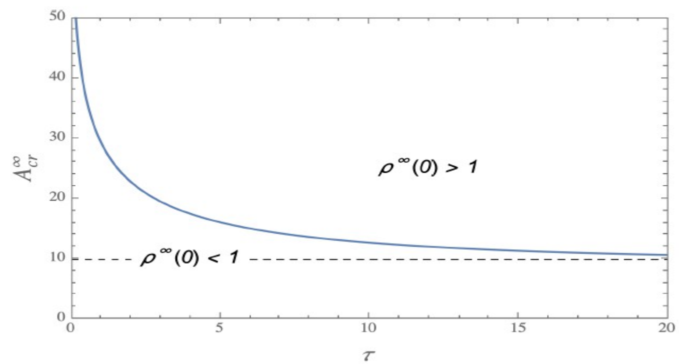
$$b_{q,cr}^{(\infty)} = \frac{1}{\left[1-\tau\left(\int dh\rho_s^{(0)}\left(A_{cr}^{(\infty)}, pb_{q,cr}^{(\infty)}, h\right)h^2 - \frac{1}{12}\right)\right]^2}$$



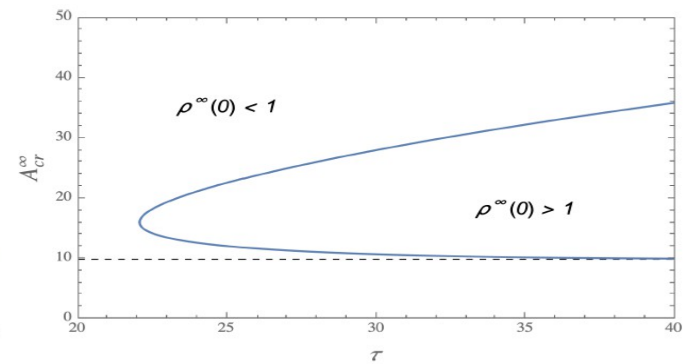
(a) $p = 2$. There is one solution $b^{(\infty)}_{q,cr} > 1$.



(b) $p = 1$. There are two solutions $b^{(\infty)}_{q,cr} > 2$.



(c) $p = 2$. Dashed line – π^2.



(d) $p = 1$. $A^{(\infty)}_{cr}$ exists for $\tau > 22.06...$

Figure 5: The solutions to the eq. (B.11) at $p = 2, 1$. The line $A^{(\infty)}_{cr}$ separates the "one-cut" phase $\rho^{(\infty)}_s(A, p, 0) < 1$ from the "double-cut" $\rho^{(\infty)}_s(A, p, 0) > 1$ one.

Conclusion

- The collective field theory for the deformed YM is derived
- The solution to the EOM in collective field theory manifests the new first order phase transition
- The critical behavior for the DK 3-rd order phase transition in deformed YM on the disc is found and the interplay between the 1-st and 3-rd order phase transitions are identified
- The critical behavior for the p=1 deformed q-YM is clarified

Open questions

- TT-deformation of the Benjamen-Ono equation. It serves as large N limit of Calogero system(Abanov,Wiegmann)(Pavshinkin, A.G in progress)
- Generalization of the fermionic language and corresponding random walk picture
- Generalization of the relation with the partition function of charged black holes(Vafa,Ooguri,Aganagic)
- Relation with the Cardy,s et al hard rods picture