Stress tensor sector of conformal correlators and holography

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Introduction

In CFTs in d spacetime dimensions four point functions depend on $z, \bar{z}$ (the positions of operator insertions) and admit a conformal block $[\mathcal{B}_k(z,\bar{z})]$ decomposition:

$$\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_2(z,\bar{z})\mathcal{O}_1(0)\rangle = \sum_k \lambda_{\mathcal{O}_1\mathcal{O}_1\mathcal{O}_k}\lambda_{\mathcal{O}_2\mathcal{O}_2\mathcal{O}_k}\mathcal{B}_k(z,\bar{z})$$

where $\mathcal{B}_k(z,\bar{z})$ are known functions (Dolan, Osborn) which contain contributions of the primary operator $\mathcal{O}_k$ and its descendants and $\lambda_{\mathcal{O}_1\mathcal{O}_1\mathcal{O}_k}, \lambda_{\mathcal{O}_2\mathcal{O}_2\mathcal{O}_k}$ are the OPE coefficients

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\mathcal{O}_k(x_3)\rangle = \frac{\lambda_{\mathcal{O}_1\mathcal{O}_1\mathcal{O}_k}}{|x_{12}|^{2\Delta_1-\Delta_k}|x_{13}|^{\Delta_k}|x_{23}|^{\Delta_k}}$$

Introduction

Central charge C_T appears in the 2-point function of stress tensors,

$$\langle T_{\mu\nu}(x) T_{\alpha\beta}(0)\rangle = \frac{C_T}{x^{2d}}\left(\frac{1}{2}I_{\mu\alpha}I_{\nu\beta} + \frac{1}{2}I_{\mu\beta}I_{\nu\alpha} - \frac{1}{d}\delta_{\mu\nu}\delta_{\alpha\beta}\right)$$

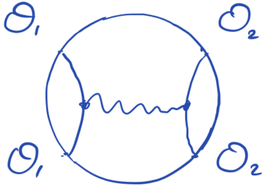
where $I_{\mu\nu} = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$. We will consider large-N CFTs with $N^2 \sim C_T \gg 1$.

Major players are double-trace operators, $[\mathcal{O}\mathcal{O}]_{n,\ell} \simeq \mathcal{O}\partial^\ell \Box^n \mathcal{O}$. For unit-normalized operators,

$$\lambda_{\mathcal{O}_1\mathcal{O}_1[\mathcal{O}_1\mathcal{O}_1]_{n,\ell}} \sim 1, \qquad \lambda_{\mathcal{O}_1\mathcal{O}_1[\mathcal{O}_2\mathcal{O}_2]_{n,\ell}} \sim \frac{1}{C_T}$$

Introduction

In holographic CFTs the leading $1/C_T \sim 1/N^2$ connected four-point function receives a contribution from a Witten diagram with the graviton exchange.



Introduction

This diagram admits a conformal block decomposition, starting with the (unit-normalized) stress tensor, $T_{\mu\nu}$. The OPE coefficient $\lambda_{\mathcal{O}\mathcal{O}T_{\mu\nu}} \sim \frac{\Delta}{\sqrt{C_T}}$ is fixed by a Ward identify, so the stress-tensor contributes $\Delta_1\Delta_2/C_T \sim 1/N^2$.

In addition, there are conformal blocks of double trace operators $[\mathcal{O}_1\mathcal{O}_1]_{n,l}$ and $[\mathcal{O}_2\mathcal{O}_2]_{n,l}$, with known OPE coefficients, which also contribute

$$\lambda_{\mathcal{O}_1\mathcal{O}_1[\mathcal{O}_1\mathcal{O}_1]_{n,l}}\lambda_{\mathcal{O}_1\mathcal{O}_1[\mathcal{O}_2\mathcal{O}_2]_{n,l}} \sim \frac{1}{C_T} \sim \frac{1}{N^2}$$

(Hijano, Kraus, Perlmutter, Snively)

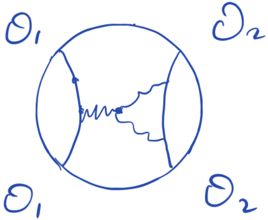
Introduction

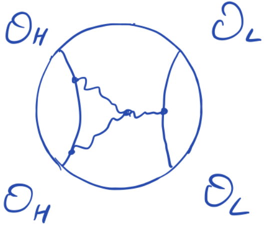
Application: Fourier transform wrt the positions of insertions; the result, $e^{i\delta}$, is not sensitive to the contribution of the double trace operators in the Regge limit (large momentum, which translates to $z, \bar{z} \to 1$, fixed $\frac{1-z}{1-\bar{z}}$). (Cornalba, Costa, Penedones, Schiappa)

Considering $T_{\mu\nu}$ as one pair of external operators (the other pair is an arbitrary scalar) and imposing positivity of the eikonal phase δ leads to the "$a = c$" condition. (equivalently, dual gravity must be Einstein-Hilbert gravity, generic higher derivative terms not allowed). (Camanho, Edelstein, Maldacena,Zhiboedov; Afkhami-Jeddi, Hartman, Kundu, Tajdini; Kulaxizi, AP, Zhiboedov; Li, Meltzer, Poland; Costa, Hansen, Penedones; Belin, Hofman, Mathys; Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov)

Introduction

1-loop diagrams are further suppressed by $1/N^2 \sim 1/C_T$. Graviton loops contribute powers of the ratio of Schwarzschild radius to the impact parameter, $(R_s/b)^k$, to δ. $R_s/b \sim 1$ - black hole creation.
(Amati, Ciafaloni, Veneziano)





What if we make one of the operators heavy, $\Delta_1 = \Delta_H \sim C_T$, keeping $\mu \simeq \Delta_H/C_T$ fixed and $\Delta_2 = \Delta_L = \mathcal{O}(1)$?

Introduction

AdS-Schwarzschild background:

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2_{d-1},$$

where

$$f = 1 + \frac{r^2}{R^2} - \frac{\mu R^{d-2}}{r^{d-2}}$$

and

$$\mu \simeq \frac{G_N M}{R^{d-2}} \simeq \frac{\Delta_H}{C_T}$$

(by AdS/CFT dictionary $\Delta_H \simeq MR$ and $R^{d-1}/G_N \simeq C_T$). So $\Delta_H \sim C_T$ is mapped to a dual black hole.

Introduction

In the CFT language, the stress-tensor contribution becomes

$$\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} \lambda_{\mathcal{O}_L \mathcal{O}_L T_{\mu\nu}} \sim \frac{\Delta_H}{\sqrt{C_T}} \frac{\Delta_L}{\sqrt{C_T}} \sim \mu$$

Double stress tensor contributions scale like

$$\lambda_{\mathcal{O}_H \mathcal{O}_H [TT]_{n,\ell}} \lambda_{\mathcal{O}_L \mathcal{O}_L [TT]_{n,\ell}} \sim \frac{\Delta_H^2}{C_T} \frac{\Delta_L^2}{C_T} \sim \mu^2$$

and k-stress tensor $[T_{\mu\nu}]^k$ contributes μ^k.

Introduction

Will be interested in the contribution of all operators $[T_{\mu\nu}]^k$ – the *stress tensor sector* of correlators. [Note: in $d = 2$ it goes under the name "the Virasoro vacuum block" and has been explicitly computed (Fitzpatrick, Kaplan, Walters).]

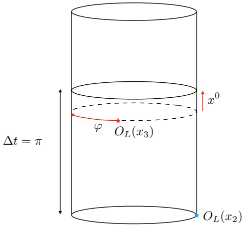
What about other operators (double trace, other light operators) ?

Few light operators in holographic theories (unless protected). Can decouple double trace operators by taking $\Delta_L \gg 1$. Also, some observables (like δ) are insensitive to double traces.

Outline

Double scaling lightcone limit

Correlator $\langle \mathcal{O}_H(t=+\infty)\,\mathcal{O}_L(\Delta t,\Delta\varphi)\,\mathcal{O}_L(0,0)\,\mathcal{O}_H(t=-\infty)\rangle$ can be viewed as 2-point function $\langle \mathcal{O}_L \mathcal{O}_L\rangle_{\mathcal{O}_H}$ in the state created by $\mathcal{O}_H$ at $t=\pm\infty$.



Cross ratios $z = e^{i(\frac{\Delta t}{R}+\Delta\varphi)}$, $\bar{z} = e^{i(\frac{\Delta t}{R}-\Delta\varphi)}$.

Double scaling lightcone limit

The stress-tensor exchange $\mathcal{O}_H\mathcal{O}_H - T_{\mu\nu} - \mathcal{O}_L\mathcal{O}_L$ produces $\mathcal{O}(\mu)$ term fixed by Ward identity; $\tau_1 = \ell(T_{\mu\nu}) - \Delta(T_{\mu\nu}) = d-2$ (minimal twist for conserved current).

Correlator at $\mathcal{O}(\mu^2)$ is a result of an infinite sum over *double stress tensor operators*: $T_{\mu\nu}\partial_\alpha \ldots \partial_\beta \Box^n T_{\gamma\delta}$.

$$T_{\mu\nu}\partial_{\mu_1}\ldots\partial_{\mu_s}T_{\alpha\beta},\ \tau_2 = 2(d-2);\ \text{leading twist}$$
$$T_\mu^{\ \alpha}T_{\alpha\nu},\ T_{\mu\nu}\partial_{\mu_1}\ldots\partial_{\mu_s}\Box\ T_{\alpha\beta}, \tau = 2(d-2)+2$$
$$T_{\mu\nu}T^{\mu\nu},\ T_\mu^{\ \alpha}\Box T_{\alpha\nu},\ T_{\mu\nu}\partial_{\mu_1}\ldots\partial_{\mu_s}\Box^2\ T_{\alpha\beta}, \tau = 2(d-2)+4$$

$\mathcal{O}(\mu^k)$ comes from k-stress tensors with leading twist $\tau_k = k(d-2)$.

Double scaling lightcone limit

To compute the stress tensor sector, need to know all OPE coefficients. Things simplify in the lightcone limit $\bar{z} \to 1$. Lightcone OPE is dominated by lowest twist:

$$\mathcal{O}_1(x)\mathcal{O}_1(0) \sim x^{-2\Delta_1} \sum \frac{x^{\mu_1} \dots x^{\mu_\ell}}{(x^2)^{\frac{\tau}{2}}} \mathcal{O}_{\mu_1 \dots \mu_\ell}$$

This is reflected in the near-lightcone behavior of $\mathcal{B}_{[T_{\mu\nu}]^k} \sim (1-\bar{z})^{k\frac{d-2}{2}}$ for leading twist $[T_{\mu\nu}]^k$. Their OPE coefficients are not affected by higher order gravitational terms (Fitzpatrick, Huang). Can keep all leading twist $[T_{\mu\nu}]^k$ by keeping $\tilde{\mu} = \mu\,(1-\bar{z})^{\frac{d-2}{2}}$ fixed.

Bootstrapping the near-lightcone correlator

Ansatz + bootstrap in $d = 4$ (any even d works similarly)

$$\mathcal{O}(\tilde{\mu}): \quad \langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(1)} \simeq [(1-z)(1-\bar{z})]^{-\Delta_L} f_3(z)$$

where

$$f_a(z) = (1-z)^a \ {}_2F_1(a, a, 2a, 1-z)$$

At higher orders, make an educated guess:

$$\mathcal{O}(\tilde{\mu}^2): \quad \langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(2)} \sim b_{33} f_3^2 + b_{24} f_2 f_4 + b_{15} f_1 f_5$$

$$\mathcal{O}(\tilde{\mu}^3): \quad \langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(3)} \sim b_{333} f_3^3 + b_{234} f_2 f_3 f_4 + \dots$$

and similarly for all values of k for $\mathcal{O}(\tilde{\mu}^k)$ coefficients. Can bootstrap and solve for all b_i's, order by order. (Karlsson, Kulaxizi, AP, Tadic)

Bootstrapping the near-lightcone correlator

Consider stress-tensor contribution, $k = 1$ term in the $\mathcal{O}_H\mathcal{O}_H - \mathcal{O}_L\mathcal{O}_L$ channel:

$$\langle \mathcal{O}_H\mathcal{O}_H\mathcal{O}_L\mathcal{O}_L\rangle^{(1)} \sim (\alpha_0 + \alpha_1 z + \ldots) + (\beta_0 + \beta_1 z + \ldots)\log z$$

with α_i, β_i known coefficients. The cross channel, $\mathcal{O}_H\mathcal{O}_L - \mathcal{O}_H\mathcal{O}_L$, produces

$$\langle \mathcal{O}_H\mathcal{O}_H\mathcal{O}_L\mathcal{O}_L\rangle^{(1)} \sim (P_0^{(1)} + P_1^{(1)} z + \ldots) + (\gamma_0^{(1)} + \gamma_1^{(1)} z + \ldots)\log z$$

where $\gamma(n,\ell) = \gamma_n^{(1)}/\ell + \mu^2\gamma_n^{(2)}/\ell^2 + \ldots$ and similarly $P(n,\ell)$ are anomalous dimensions and OPE coefficients of heavy-light double trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,\ell}$.

Bootstrapping the near-lightcone correlator

This determines $P_n^{(1)}, \gamma_n^{(1)}$ and all terms $\sim \log^2 z$ at $\mathcal{O}(\tilde{\mu}^2)$. This determines **all** coefficients in the ansatz at $\mathcal{O}(\tilde{\mu}^2)$:

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle^{(2)} \simeq \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{\Delta_L - 2} \right) \times \left[(\Delta_L - 4)(\Delta_L - 3) f_3(z)^2 + \frac{15}{7}(\Delta_L - 8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L + 1) f_1(z) f_5(z) \right]$$

(Kulaxizi, Ng, AP)

Bootstrapping the near-lightcone correlator

OPE coefficients with the leading twist double-stress operators:

$$\lambda_{\mathcal{O}_H\mathcal{O}_H[T^2_{\mu\nu}]_{n=0,\ell}}\lambda_{\mathcal{O}_L\mathcal{O}_L[T^2_{\mu\nu}]_{n=0,\ell}} = \frac{{\Delta_H}^2}{C_T}\,\frac{a_\ell^2}{C_T}\,\frac{\Delta_L}{\Delta_L-2}\,({\Delta_L}^2+b_\ell\Delta_L+c_\ell)$$

where

$$b_\ell = -1 + \frac{36}{\ell(\ell+3)+c_\ell}, \quad c_\ell = \frac{288}{(\ell-2)\ell(\ell+3)(\ell+5)}, a_\ell = \dots$$

so

$$\lambda_{\mathcal{O}_L\mathcal{O}_L[T^2_{\mu\nu}]_{n=0,\ell}} = \frac{a_\ell}{C_T}\,\frac{\Delta_L}{\Delta_L-2}\,({\Delta_L}^2+b_\ell\Delta_L+c_\ell)$$

(Kulaxizi, Ng, AP)

Bootstrapping the near-lightcone correlator

One can continue this procedure and compute the lightcone correlator to any desired order in $\tilde{\mu}$

We can read off the OPE coefficients $\lambda_{\mathcal{O}_L\mathcal{O}_L[T^k_{\mu\nu}]_{n=0,\ell}}$ to leading order in $1/C_T$. We did not use holography, but recover the OPE coefficients which have been computed using holography (Fitzpatrick, Huang) (and get many more).

The correlator exponentiates

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle \sim e^{\Delta_L \mathcal{F}(z,\ \tilde{\mu},\ \Delta_L)}$$

where $\mathcal{F}(z,\ \tilde{\mu},\ \Delta_L)$ has a finite limit $\mathcal{F}_\infty$ as $\Delta_L \to \infty$.

Bootstrapping the near-lightcone correlator

The result further simplifies in the large volume limit ($R \to \infty$). Only operators $T_{\mu\nu} \dots T_{\alpha\beta}$ contribute (no derivatives allowed) and the result is an expansion in powers of $\mu\, \Delta x^-(\Delta x^+)^3$:

$$\mathcal{F}_\infty|_{d=4} = -\log(\Delta x^+ \Delta x^-) + \frac{\mu \Delta x^-(\Delta x^+)^3}{120} + \frac{\mu^2(\Delta x^-)^2(\Delta x^+)^6}{10080} + \frac{1583\mu^3(\Delta x^-)^3(\Delta x^+)^9}{648648000} + \dots,$$

Holographic leading twist calculation

In holographic CFTs we can compute

$$\langle \mathcal{O}_H(t=+\infty)\mathcal{O}_L(\Delta t,\Delta\varphi)\mathcal{O}_L(0,0)\mathcal{O}_H(t=-\infty)\rangle$$

for $\Delta_L \gg 1$ by computing

$$\langle \mathcal{O}_L(\Delta t,\Delta\varphi)\mathcal{O}_L(0,0)\rangle = e^{-\Delta_L \ell}$$

where $\ell = -\mathcal{F}_\infty$ is the regularized length of the geodesic which connects points $(\Delta t, \Delta\varphi)$ and $(0,0)$ on the boundary of AdS-Schwarzschild spacetime.

Holographic leading twist calculation

Rescale coordinates in the AdS-Schwarzschild metric,
$x^- = (t-\varphi)\mu^{\frac{2}{d-2}}$ and $y = r\mu^{-\frac{1}{d-2}}$ and consider the $\mu\to\infty$ limit keeping $x^\pm$, y fixed (here $R=1$)

$$ds^2 = -\frac{1}{4}\left(1-\frac{1}{y^{d-2}}\right)(dx^+)^2 - y^2 dx^+\, dx^- + \frac{dy^2}{y^2}.$$

Two Killing vectors, ∂_+ and ∂_- give rise to two conserved quantities, K and K_+. Geodesic equation (spacelike) becomes

$$\dot{y}^2 + 4KK_+ + (y^{-2} - y^{-d})K^2 - y^2 = 0.$$

Holographic leading twist calculation

Can find the length of the geodesic $\ell(\Delta x^+, \Delta x^-)$. $\mathcal{F}_\infty = -\ell$ but it's UV-divergent, need to regularize. Things simplify in the large volume limit.

The solution can be written in terms of elliptic integrals [Appel functions]

$$\ell_f \simeq \log(\Delta x^+ \Delta x^-) - \log[I_+(\alpha) I_-(\alpha)] + I_\ell(\alpha)$$

where α is determined by

$$(-\Delta x^-)^{\frac{d-2}{2}} (\Delta x^+)^{\frac{d+2}{2}} = \alpha \, I_-^{\frac{d-2}{2}}(\alpha) \, I_+^{\frac{d+2}{2}}(\alpha)$$

and

$$I_\ell(x) = 2 \int_{u_0}^{\Lambda_u} \frac{u^{\frac{d}{2}} du}{(u^{d+2} - 4u^d + x)^{\frac{1}{2}}} - 2 \log \Lambda_u$$

Holographic leading twist calculation

$d = 2$: recover HHLL Virasoro block:

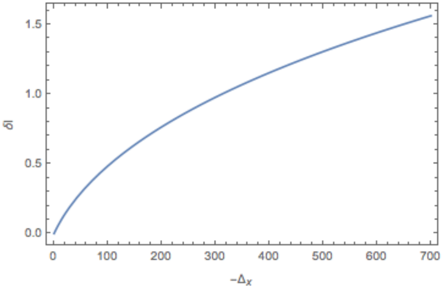
$$-\ell_f\big|_{d=2} \simeq -\log\sinh\frac{\sqrt{\mu}\Delta x^+}{2},$$

$d = 4$: new result; expansion in $\Delta_x = \mu\Delta x^-(\Delta x^+)^3 \simeq T^4\Delta x^-(\Delta x^+)^3$ (where T is the temperature). Agrees with bootstrap:

$$-\ell_f\big|_{d=4} \simeq -\log(\Delta x^-\Delta x^+)+\frac{\Delta_x}{120}+\frac{\Delta_x^2}{10080}+\frac{1583\Delta_x^3}{648648000} + \frac{3975313\Delta_x^4}{49401031680000}+\ldots.$$

(AP)

Holographic leading twist calculation



$\mathcal{F}_\infty|_{d=4}$ with $\log(\Delta x^+ \Delta x^-)$ term subtracted as a function of $\mu \Delta x^- (\Delta x^+)^3$. (AP)

Beyond leading twist

Bootstrap can be successfully extended beyond the leading twist. (Karlsson, Kulaxizi, AP, Tadic)

The results agree with the phase shift (a.k.a. eikonal phase) δ, which has been explicitly computed in gravity to all orders in the impact parameter for the HHLL case. (Kulaxizi, Ng, AP; AP, Sen)

What happens in non-holographic theories? There are finite gap corrections to $\lambda_{\mathcal{O}\mathcal{O}[T_{\mu\nu}]^k}$ (Fitzpatrick, Huang, Meltzer, Perlmutter, Simmons-Duffin). Is the stress tensor sector still special in any way?

Thermalization of stress-tensor sector

In large-N CFTs stress-tensor thermalizes,

$$\langle \mathcal{O}_H | T_{\mu\nu} | \mathcal{O}_H \rangle = \langle T_{\mu\nu} \rangle_\beta$$

where β is determined by the saddle-point equation, $\Delta_H / R = \partial_\beta (\beta F)$. Large-N factorization implies $\langle [T_{\mu\nu}]^k \rangle_\beta = \langle T_{\mu\nu} \rangle_\beta^k \simeq {\Delta_H}^k$. Hence, thermalization of $[T_{\mu\nu}]^k$ is equivalent to

$$\langle \mathcal{O}_H | [T_{\mu\nu}]^k | \mathcal{O}_H \rangle = \# {\Delta_H}^k \left[1 + \mathcal{O}(\frac{1}{C_T}) \right]$$

This is a statement about the leading Δ behavior of multi-stress tensor OPE coefficients, which can be verified. (Karlsson, AP, Tadic)

Summary

- Can compute HHLL correlator to all orders in μ via bootstrap.
- Computed it in the large Δ_L limit in holography – perfect agreement.
- Stress tensor sector thermalizes (even in non-holographic large N CFTs).

To do list

- Find closed form of heavy-heavy-light-light correlator
- Symmetry of the lightcone correlator?
- Understand thermal strongly coupled large-N CFTs
- Regge scattering in gravity

Thank you!