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Integrability, Holography, Higher-Spin Gravity and Strings

Observables and Invariants in 4D Higher Spin Gravity

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General Remarks

Lower-Spin Success Stories

Spin-0, 1/2, 1:

Standard Model $\leftrightarrow$ Geometry of vector bundles/Semisimple Lie groups and their reps.

Spin-2:

Einstein Gravity $\leftrightarrow$ Reimann geometry/Tensor algebra

Spin > 2:

HS Gravity $\leftrightarrow$ Non-commutative geometry/Cyclic cohomology

(Euclidean Geometry) $\xrightarrow{\text{GR}}$ (Non-Euclidean Geometry) $\xrightarrow{\text{HSGRA}}$ (Non-Commutative Geometry)

Outline

- Formal dynamical systems and their invariants
- Higher-spin algebra in $D = 4$
- Classification of invariants in $4D$ HSGRA
- Weak Lagrangians for $4D$ HSGRA
- Higher-spin waves and currents

Formal Dynamical Systems

Let $\{W^A\}$ be a collection of differential forms on M, then EoM read

$$dW^A = f^A_{BC} W^A \wedge W^B + c^A_{BCD} W^B \wedge W^C \wedge W^D + \cdots \equiv Q^A(W)$$

[D. Sullivan; R. D'Auria & P. Fré; P. van Nieuwenhuizen; M. Vasiliev, ...]

Formal integrability ($d^2 = 0 \;\Rightarrow\; Q^2 = 0$) implies:

- f^A_{BC} are structure constants of some graded Lie algebra L;
- c^A_{BCD} is a Chevalley–Eilenberg cocycle of the Lie algebra L.

Together the f's, $c's$, ... define the structure of an L_∞-algebra.

The system enjoys the gauge symmetry

$$\delta_\varepsilon W^A = d\varepsilon^A + \varepsilon^B \wedge \partial_B Q^A(W)$$

Physical Observables

$$\mathcal{Q} = \int_\Sigma J\,, \qquad J = J_n + J_{n+1} + J_{n+2} + \cdots\,, \qquad \partial\Sigma = 0\,,$$

$$J_n = J_{A_1 A_2 \cdots A_n} W^{A_1} \wedge W^{A_2} \wedge \cdots \wedge W^{A_n}\,.$$

Gauge invariance:

$$\delta_\varepsilon \mathcal{Q} \approx 0 \quad \Leftrightarrow \quad dJ \approx 0 \quad \text{(on-shell)}$$

J defines a (lower-degree) conservation law (aka characteristic cohomology).

$J_{A_1 \cdots A_n}$ is a scalar cocycle of the Lie algebra L:

$$f^A_{[A_0 A_1} J_{A A_2 \cdots A_n]} = 0\,.$$

[G. Barnich & M. Grigoriev, 2011]

Higher Spin Algebras

Fact: The Lie algebras underlying HSGRA originate from **associative algebras**:

$$L = L(A)\,, \qquad [a,b] = ab - (-1)^{|a||b|} ba\,, \quad \forall a, b \in A\,.$$

AdS/CFT

HSGRA $\longleftrightarrow$ Free CFT

$\Downarrow$

$\Uparrow$ Inf. assoc. alg. of symmetries

$\Downarrow$

Gauge fields $\longleftrightarrow$ HS currents $J_s = \phi \partial^s \phi$

[B. Sundborg; E. Sezgin & P. Sundell; I. Klebanov & A. Polyakov]

From Chevalley–Eilenberg to Hochschild

Let $gl(A)$ be the matrix extension of a HS algebra A.

Chevalley–Eilenberg	$\Rightarrow$	Cyclic	$\Rightarrow$	Hochschild
$H^\bullet(gl(A))$		$HC^\bullet(A)$		$HH^\bullet(A)$
		$\Uparrow$		
		Non-Commutative Geometry		

4D Higher Spin Gravity

Fields: the 1-form field ω and 0-form field C with values in $gl(A)$.

Field equations:

$$d\omega = \omega \wedge \omega + \mathcal{V}(\omega, \omega, C) + \cdots \,, \qquad dC = \omega C - C\omega + \mathcal{V}(\omega, C, C) + \cdots$$

The extended HS algebra in 4D [E. Fradkin & M. Vasiliev, '87]:

$$A = (A_1 \rtimes \mathbb{Z}_2) \otimes (A_1 \rtimes \mathbb{Z}_2)\,,$$

$$A_1 \rtimes \mathbb{Z}_2 : \quad qp - pq = 1\,, \qquad \kappa q = -q\kappa\,, \qquad \kappa p = -p\kappa\,, \qquad \kappa^2 = 1\,.$$

$HH^2(A, A) = \mathbb{R}^2 \quad \Rightarrow \quad$ 2-parameter deformation $A(\nu, \bar{\nu}) \quad \Rightarrow \quad$ 2 coupling constants:

$$qp - pq = 1 + \nu\kappa \quad \text{(deformed oscillator algebra)}$$

[E. Wigner, '50]

Classification of Observables in 4D HSGRA

$$J_{n,m} = J(\underbrace{\omega,\ldots,\omega}_{n},\underbrace{C,\ldots,C}_{m}) + o(C^{m+1})\,, \qquad dJ_{n,m} \approx 0\,, \qquad n = 0,1,2,3,4.$$

- 'Holographic correlators': $J_{0,n} = \mathrm{Tr}(C^n) + \cdots$
- Surface currents: $J_{2,2n+1} = \mathrm{Tr}(\mathcal{V}_{1,2}(\omega,\omega,C)C^{2n}) + \cdots$
- Counter-terms: $J_{4,2n+2} = \mathrm{Tr}\big(\mathcal{V}_1(\omega,\omega,C)\mathcal{V}_2(\omega,\omega,C)C^{2n}\big) + \cdots$

$\nexists$ gauge invariants $\sim \omega^4 + \cdots$ **but** $\exists$ gauge invariants $\sim \omega^5 + \cdots$

[E. Sezgin, P. Sundell, C. Iazeolla, N. Colombo, V. Didenko, E. Skvortsov, ...]
[V. Didenko, N. Misuna, M. Vasiliev, 2015]
[M. Vasiliev, 2015]

Presymplectic AKSZ Models

The Lagrangian of an AKSZ-type σ-model reads

$$\mathcal{L} = \Theta_A(W) \wedge dW^A - H(W)\,.$$

Geometrically,

- $H(W)$ is a function,
- $\Theta = \Theta_A(W)\delta W^A$ is the 1-form of presymplectic potential, and
- $\Omega = \delta\Theta$ is a presymplectic 2-form on the (graded) target space of W's.

$$\delta\mathcal{L} = 0 \quad \Leftrightarrow \quad \Omega_{AB}\big(dW^A - Q^A(W)\big) = 0\,,$$

$Q^A\Omega_{AB} = \partial_B H$, i.e. Q is a Hamiltonian vector field associated with H.

If (Ω_{AB}) is degenerate, then $\mathcal{L}$ is a **weak Lagrangian** for $dW^A = Q^A(W)$.

[K. Alkalaev & M. Grigoriev, 2014]

Weak Lagrangians for 4D HSGRA

$$\mathcal{L}_t = \mathrm{Tr}\Big[\mathcal{V}_t(\omega,\omega,C)\wedge(d\omega-\omega\wedge\omega)\Big] + o(C^2)\,,\qquad \mathcal{V}_t = \cos(t)\mathcal{V}_1+\sin(t)\mathcal{V}_2$$

$$EL(\mathcal{L}_t) \supset \text{(solutions to HSGRA EoM)}$$

The leading term in $\mathcal{L}_t$ provides the action principle for the 'free EoM'

$$d\omega = \omega\wedge\omega\,,\qquad dC = [\omega, C]\,.$$

$\mathcal{V}_t(\omega,\omega,C)$ is an **integrating multiplier** of the inverse problem of calculus of variations.

[K. Krasnov, E. Skvortsov, T. Tran, 2021]

Other proposal for Lagrangian: [N. Boulanger & P. Sundell, 2011].

Higher Spin Waves and Currents

Linearization over the HS vacuum $d\omega = \omega \wedge \omega$, $C = 0$ gives the EoM for HS waves:

$$D\tilde{\omega} = \mathcal{V}(\omega, \omega, \tilde{C})\,, \qquad D\tilde{C} = 0\,.$$

[C. Aragone & S. Deser '79]

- Local symmetries: $\delta_\varepsilon \tilde{\omega} = D\varepsilon\,, \qquad \delta_\varepsilon \tilde{C} = 0\,.$
- Global symmetries: $\delta_\xi \tilde{\omega} = [\xi, \tilde{\omega}] + \mathcal{V}(\xi, \omega, \tilde{C}) - \mathcal{V}(\omega, \xi, \tilde{C})\,, \quad \delta_\xi C = [\xi, \tilde{C}]\,, \quad D\xi = 0.$

The weak Lagrangian for HS waves reads

$$\tilde{\mathcal{L}}_t = \mathrm{Tr}\big[\mathcal{V}_t(\omega, \omega, \tilde{C}) \wedge D\tilde{\omega} + \Lambda_t(\omega, \omega, \omega, \tilde{C}) \wedge D\tilde{C} - \tfrac{1}{2}\mathcal{V}_t(\omega, \omega, \tilde{C}) \wedge \mathcal{V}(\omega, \omega, \tilde{C})\big].$$

Noether's correspondence $\Rightarrow$ (gauge non-invariant) HS conserved currents:

$$J_\xi = \tilde{C} \times \tilde{C} + \tilde{C} \times \tilde{\omega}\,, \qquad \delta_\varepsilon J_\xi = d(\ldots)\,.$$

[O. Gelfond, E. Skvotsov, M. Vasiliev, 2008; P. Smirnov & M. Vasiliev, 2017]

Where does Non-Commutative Geometry Come From?

$$dW^A = Q^A(W)\,, \qquad Q^2 = 0\,.$$

Algebra | **Geometry**

L_∞-algebra, i.e. ∞-extension of the graded Lie bracket [a,b]$=\pm[b,a]$

$\Longleftrightarrow$

Homological vector field Q on a graded manifold $\mathcal{M}$ with coordinates $W^A W^B = \pm W^B W^A$

antisymmetrization ↑ | symmetrization ↑

A_∞-algebra, i.e. ∞-extension of an associative algebra structure ab$\neq \pm ba$

$\Longleftrightarrow$

Homological vector field Q on a **non-commutative** manifold $\mathcal{M}$ with coordinates $W^A W^B \neq \pm W^B W^A$

Conclusion

We constructed and classified all physical observables, presymplectic structures and weak Lagrangians for 4D HSGRA.

Further perspectives:

- Deformation/path-integral quantization of HSGRA.
- Renormalizability/finiteness of HSGRA (no local counter-terms).
- Correlation functions in Chern–Simons Matter theories / 3D bosonization.

Thank You !