

QUARKS ONLINE WORKSHOPS-2021, May 31 – June 5, 2021

Integrability, Holography, Higher-Spin Gravity and Strings

Observables and Invariants in 4D Higher Spin Gravity

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[arXiv:2102.02253](https://arxiv.org/abs/2102.02253), [arXiv:2006.13986](https://arxiv.org/abs/2006.13986)

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General Remarks

Lower-Spin Success Stories

Spin-0, 1/2, 1:

Standard Model \leftrightarrow Geometry of vector bundles/Semisimple Lie groups and their reps.

Spin-2:

Einstein Gravity \leftrightarrow Riemann geometry/Tensor algebra

Spin $>$ 2:

HS Gravity \leftrightarrow Non-commutative geometry/Cyclic cohomology

(Euclidean Geometry) $\xrightarrow{\text{GR}}$ (Non-Euclidean Geometry) $\xrightarrow{\text{HSGRA}}$ (Non-Commutative Geometry)

Outline

- Formal dynamical systems and their invariants
- Higher-spin algebra in $D = 4$
- Classification of invariants in $4D$ HSGRA
- Weak Lagrangians for $4D$ HSGRA
- Higher-spin waves and currents

Formal Dynamical Systems

Let $\{W^A\}$ be a collection of differential forms on M , then EoM read

$$dW^A = f_{BC}^A W^A \wedge W^B + c_{BCD}^A W^B \wedge W^C \wedge W^D + \dots \equiv Q^A(W)$$

[D. Sullivan; R. D'Auria & P. Fré; P. van Nieuwenhuizen; M. Vasiliev, ...]

Formal integrability ($d^2 = 0 \Rightarrow Q^2 = 0$) implies:

- f_{BC}^A are structure constants of some graded Lie algebra L ;
- c_{BCD}^A is a Chevalley–Eilenberg cocycle of the Lie algebra L .

Together the f 's, c 's, ... define the structure of an L_∞ -algebra.

The system enjoys the gauge symmetry

$$\delta_\varepsilon W^A = d\varepsilon^A + \varepsilon^B \wedge \partial_B Q^A(W)$$

Physical Observables

$$Q = \int_{\Sigma} J, \quad J = J_n + J_{n+1} + J_{n+2} + \dots, \quad \partial\Sigma = 0,$$

$$J_n = J_{A_1 A_2 \dots A_n} W^{A_1} \wedge W^{A_2} \wedge \dots \wedge W^{A_n}.$$

Gauge invariance:

$$\delta_{\varepsilon} Q \approx 0 \quad \Leftrightarrow \quad dJ \approx 0 \quad (\text{on-shell})$$

J defines a (lower-degree) conservation law (aka [characteristic cohomology](#)).

$J_{A_1 \dots A_n}$ is a scalar cocycle of the Lie algebra L :

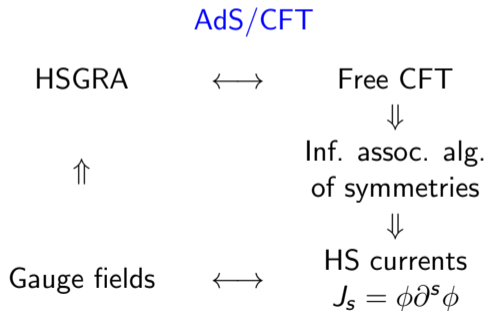
$$f_{[A_0 A_1}^{A} J_{A A_2 \dots A_n]} = 0.$$

[G. Barnich & M. Grigoriev, 2011]

Higher Spin Algebras

Fact: The Lie algebras underlying HSGRA originate from **associative algebras**:

$$L = L(A), \quad [a, b] = ab - (-1)^{|a||b|}ba, \quad \forall a, b \in A.$$



[B. Sundborg; E. Sezgin & P. Sundell; I. Klebanov & A. Polyakov]

From Chevalley–Eilenberg to Hochschild

Let $gl(A)$ be the matrix extension of a HS algebra A .

Chevalley–Eilenberg

$H^\bullet(gl(A))$

\Rightarrow

Cyclic

$HC^\bullet(A)$

\Rightarrow

Hochschild

$HH^\bullet(A)$

\Uparrow

Non-Commutative Geometry

4D Higher Spin Gravity

Fields: the 1-form field ω and 0-form field C with values in $gl(A)$.

Field equations:

$$d\omega = \omega \wedge \omega + \mathcal{V}(\omega, \omega, C) + \dots, \quad dC = \omega C - C\omega + \mathcal{V}(\omega, C, C) + \dots$$

The extended HS algebra in 4D [E. Fradkin & M. Vasiliev, '87]:

$$A = (A_1 \rtimes \mathbb{Z}_2) \otimes (A_1 \rtimes \mathbb{Z}_2),$$

$$A_1 \rtimes \mathbb{Z}_2 : \quad qp - pq = 1, \quad \kappa q = -q\kappa, \quad \kappa p = -p\kappa, \quad \kappa^2 = 1.$$

$HH^2(A, A) = \mathbb{R}^2 \Rightarrow$ 2-parameter deformation $A(\nu, \bar{\nu}) \Rightarrow$ 2 coupling constants:

$$qp - pq = 1 + \nu\kappa \quad (\text{deformed oscillator algebra})$$

[E. Wigner, '50]

Classification of Observables in 4D HSGRA

$$J_{n,m} = J(\underbrace{\omega, \dots, \omega}_n, \underbrace{C, \dots, C}_m) + o(C^{m+1}), \quad dJ_{n,m} \approx 0, \quad n = 0, 1, 2, 3, 4.$$

- 'Holographic correlators': $J_{0,n} = \text{Tr}(C^n) + \dots$
- Surface currents: $J_{2,2n+1} = \text{Tr}(\mathcal{V}_{1,2}(\omega, \omega, C)C^{2n}) + \dots$
- Counter-terms: $J_{4,2n+2} = \text{Tr}(\mathcal{V}_1(\omega, \omega, C)\mathcal{V}_2(\omega, \omega, C)C^{2n}) + \dots$

\nexists gauge invariants $\sim \omega^4 + \dots$ **but** \exists gauge invariants $\sim \omega^5 + \dots$

[E. Sezgin, P. Sundell, C. Iazeolla, N. Colombo, V. Didenko, E. Skvortsov, ...]

[V. Didenko, N. Misuna, M. Vasiliev, 2015]

[M. Vasiliev, 2015]

Presymplectic AKSZ Models

The Lagrangian of an AKSZ-type σ -model reads

$$\mathcal{L} = \Theta_A(W) \wedge dW^A - H(W).$$

Geometrically,

- $H(W)$ is a function,
- $\Theta = \Theta_A(W)\delta W^A$ is the 1-form of presymplectic potential, and
- $\Omega = \delta\Theta$ is a presymplectic 2-form on the (graded) target space of W 's.

$$\delta\mathcal{L} = 0 \quad \Leftrightarrow \quad \Omega_{AB}(dW^A - Q^A(W)) = 0,$$

$Q^A\Omega_{AB} = \partial_B H$, i.e. Q is a Hamiltonian vector field associated with H .

If (Ω_{AB}) is degenerate, then \mathcal{L} is a **weak Lagrangian** for $dW^A = Q^A(W)$.

[K. Alkalaev & M. Grigoriev, 2014]

Weak Lagrangians for 4D HSGRA

$$\mathcal{L}_t = \text{Tr} \left[\mathcal{V}_t(\omega, \omega, C) \wedge (d\omega - \omega \wedge \omega) \right] + o(C^2), \quad \mathcal{V}_t = \cos(t)\mathcal{V}_1 + \sin(t)\mathcal{V}_2$$

$$EL(\mathcal{L}_t) \supset (\text{solutions to HSGRA EoM})$$

The leading term in \mathcal{L}_t provides the action principle for the 'free EoM'

$$d\omega = \omega \wedge \omega, \quad dC = [\omega, C].$$

$\mathcal{V}_t(\omega, \omega, C)$ is an **integrating multiplier** of the inverse problem of calculus of variations.

[K. Krasnov, E. Skvortsov, T. Tran, 2021]

Other proposal for Lagrangian: [N. Boulanger & P. Sundell, 2011].

Higher Spin Waves and Currents

Linearization over the HS vacuum $d\omega = \omega \wedge \omega$, $C = 0$ gives the EoM for HS waves:

$$D\tilde{\omega} = \mathcal{V}(\omega, \omega, \tilde{C}), \quad D\tilde{C} = 0. \quad [\text{C. Aragone \& S. Deser '79}]$$

- Local symmetries: $\delta_\epsilon \tilde{\omega} = D\epsilon$, $\delta_\epsilon \tilde{C} = 0$.
- Global symmetries: $\delta_\xi \tilde{\omega} = [\xi, \tilde{\omega}] + \mathcal{V}(\xi, \omega, \tilde{C}) - \mathcal{V}(\omega, \xi, \tilde{C})$, $\delta_\xi C = [\xi, \tilde{C}]$, $D\xi = 0$.

The weak Lagrangian for HS waves reads

$$\tilde{\mathcal{L}}_t = \text{Tr}[\mathcal{V}_t(\omega, \omega, \tilde{C}) \wedge D\tilde{\omega} + \Lambda_t(\omega, \omega, \omega, \tilde{C}) \wedge D\tilde{C} - \frac{1}{2}\mathcal{V}_t(\omega, \omega, \tilde{C}) \wedge \mathcal{V}(\omega, \omega, \tilde{C})].$$

Noether's correspondence \Rightarrow (gauge non-invariant) HS conserved currents:

$$J_\xi = \tilde{C} \times \tilde{C} + \tilde{C} \times \tilde{\omega}, \quad \delta_\epsilon J_\xi = d(\dots).$$

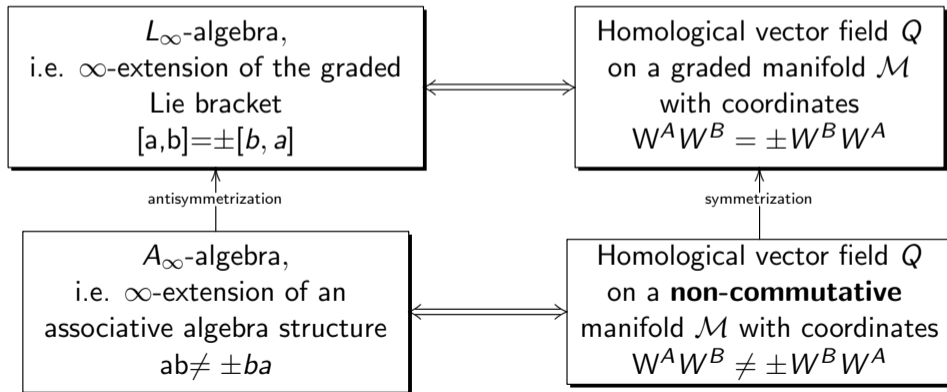
[O. Gelfond, E. Skvotsov, M. Vasiliev, 2008; P. Smirnov & M. Vasiliev, 2017]

Where does Non-Commutative Geometry Come From?

$$dW^A = Q^A(W), \quad Q^2 = 0.$$

Algebra

Geometry



Conclusion

We constructed and classified all physical observables, presymplectic structures and weak Lagrangians for 4D HSGRA.

Further perspectives:

- Deformation/path-integral quantization of HSGRA.
- Renormalizability/finiteness of HSGRA (no local counter-terms).
- Correlation functions in Chern–Simons Matter theories / 3D bosonization.

Thank You !