Higher-spin BMS algebras

Xavier BEKAERT

Institut Denis Poisson (Tours)

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joint work with Blagoje Oblak (work in progress)

Broad motivations

Despite the wide array of no-go theorems against interacting massless theories in Minkowski spacetime (of dimension 4 and higher),

- the cardinal importance of flat spacetime for physical applications
- and the old issue of string theory symmetries in the tensionless limit

can be taken as broad motivation for studying higher-spin symmetries in flat spacetime.

Higher-spin motivations

Two tantalising questions:

1. What might be an analogue of the singleton in flat spacetime?
2. What might be higher-spin symmetry algebra in flat spacetime? *

*Open question for spacetime dimension 4 and higher, but known in dimension 3 (Afshar-Bagchi-Fareghbal-Grumiller-Rosseel, 2013; Gonzalez-Matulich-Pino-Troncoso, 2013; ...)

Tools

Two main tools available:

1. BMS representation theory
   - BMS_4: Seminal works
     - Sachs (1962)
     - Series of papers by McCarthy (1972-1975)
   - BMS_3: Barnich & Oblak (2014-2015)
   - $BMS_{>4}$: ?
2. BMS intrinsic geometry
   - Seminal (Penrose, 1965)
   - Many contributions (Ashtekar, ...)
   - Modern view as conformal Carroll (Duval-Gibbons-Horvathy, 2014)

Goals

Two main goals:

1. Discuss two BMS analogues of Rac (= scalar singleton) :
   1. Wick-rotated Rac
      - + looks natural and familiar
      - – seems *not* unitarisable
      - – is *not* faithful representation of BMS (nor Poincaré) only of Lorentz
   2. Sachs representation
      - – qualitatively $\neq$ Rac (less degenerate)
      - + faithful *and* unitary representation of BMS
      - + corresponds to irrep of Poincaré (massless scalar, radiation solutions)

Goals

Two main goals:

1. Discuss two BMS analogues of Rac (= scalar singleton)
2. Construct the corresponding higher-spin extension(s) of (extended) BMS algebra(s)
   - off-shell: $\exists$
     - Contains the higher-spin extension of Poincaré algebra (XB, 2010)
     - Make contact with BMS Killing tensors obtained from the asymptotic symmetries of free massless higher-spin fields (Campoleoni-Francia-Heissenberg, 2017-2020)

     Linear structure $\Rightarrow$ Algebra structure
   - on-shell: Degenerate or $\nexists$?

Outline

1. Introduction
   - Motivations
   - Goals
   - Outline
2. Carroll geometry
   - Motivation: BMS as conformal Carroll
   - Ambient geometry
   - Carrollian structures
3. BMS as Conformal Carroll
   - Conformal Carrollian structures and symmetries
   - Wick-rotated Rac
   - Sachs representation
4. Conclusion

Intrinsic and geometric view on BMS symmetries

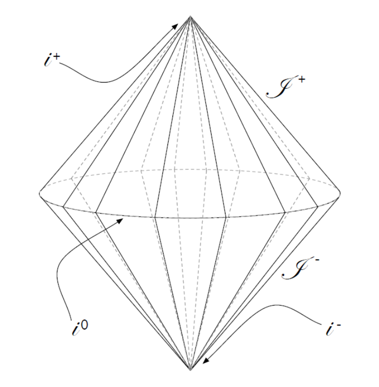


Fig. 1 in [J.-P. Nicolas, `arXiv:1508.02592 [hep-th]`]

Intrinsic view

Although BMS group is most often discussed from the point of view of asymptotic symmetries of a bulk spacetime, it can be formulated in an

- intrinsic (*i.e.* purely from the boundary) and
- geometric (*i.e.* global and coordinate-free) way.

This point of view on BMS group

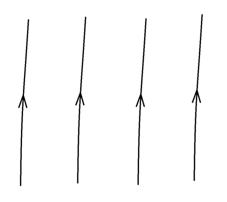
- goes back to Penrose (1965)
- allows to interpret BMS group as a conformal extension of Carroll group (Duval-Gibbons-Horvathy, 2014)

Ambient geometry

Ambient structures

Ambient structure: The following geometrical structures are equivalent

- Nowhere vanishing vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on a manifold $\mathscr{M}$
- Congruence of parametrised curves from $\mathbb{R}$ to $\mathscr{M}$
- Principal $\mathbb{R}$-bundle $\mathscr{M}$ with fundamental vector field ξ



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- The curves are the integral lines of the fundamental vector field; they are also the orbits of the $\mathbb{R}$-action on $\mathscr{M}$.
- The space $\bar{\mathscr{M}}$ of such orbits is the base manifold of the principal bundle

$$\bar{\mathscr{M}} = \mathscr{M}/\mathbb{R}$$

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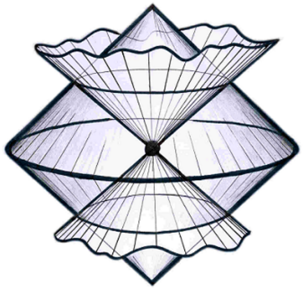
Local expression: there exist a coordinate system (u, x^i) such that

- Fundamental vector field $\xi = \frac{\partial}{\partial u}$
- Curves $x^i = x^i_0$ parametrised by u
- $\mathbb{R}$-action $u \to u - u_0$ $(u_0 \in \mathbb{R})$
- Fibration $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}} : (u, x^i) \mapsto x^i$

Ambient structures

Example 1: Möbius model (projective null cone)

- Past lightcone $\mathscr{N}^- \subset \mathbb{R}^{d+1,1}$ of the origin of Minkowski spacetime
- Coordinates (u, θ^i) on $\mathscr{N}^- \cong \mathbb{R} \times S^d$
- Fundamental vector field $\xi = \frac{\partial}{\partial u}$ is null
- Null rays generating the cone
- $\mathbb{R}$-action $u \to u - u_0$ $(u_0 \in \mathbb{R})$
- Fibration $\pi : \mathscr{N}^- \twoheadrightarrow S^d : (u, \theta^i) \mapsto \theta^i$

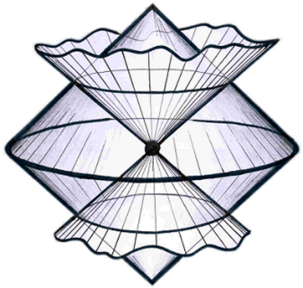


Cover of [B.-Oblak, arXiv:1610.08526 [hep-th]]

Ambient structures

Example 2: Inversion $x^\mu = \frac{x^\mu}{x^2} \quad \Rightarrow \quad \mathscr{N}^\mp \leftrightarrow \mathscr{I}^\pm$

- Future null infinity $\mathscr{I}^+$ at the conformal boundary of compactified Minkowski spacetime
- Coordinates (u, θ^i) on $\mathscr{I}^+ \cong \mathbb{R} \times S^d$
- Etc (idem as $\mathscr{N}^-$) $\cdots$



Cover of [B.-Oblak, arXiv:1610.08526 [hep-th]]

Projection on the base manifold

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$ with fundamental vector field ξ.

- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$
- **Projectable vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = f\,\xi$ for some $f \in C^\infty_{\text{inv}}(\mathscr{M})$
- **Invariant metric tensor:** $\mathcal{L}_\xi \gamma_{\mu\nu} = 0$
- **Projectable metric tensor:** $\mathcal{L}_\xi \gamma_{\mu\nu} = 0$ and $\gamma_{\mu\nu}\xi^\nu = 0$

Intrinsic view on Carroll geometry

Carrollian structure

Fundamental vector field

&

Carrollian metric

Timelike metric structure

Fundamental vector field: Nowhere vanishing vector field on the spacetime manifold, $\xi = \xi^\mu \partial_\mu \neq 0$

Provides a distinction between the type of vectors in Carroll geometry:

$$\begin{cases} V^\mu = f\,\xi^\mu & \text{with} \quad \begin{cases} f \neq 0 & \textbf{Timelike} \text{ (or } \textbf{Vertical}\text{)} \\ f > 0 & \textbf{Future-oriented} \end{cases} \\ V^\mu \neq f\,\xi^\mu & \qquad\qquad \textbf{Spacelike} \end{cases}$$

Timelike metric structure

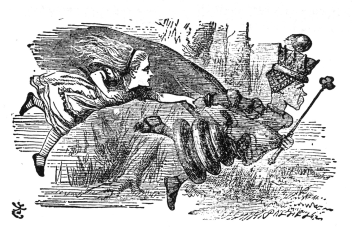
Fundamental vector field: Nowhere vanishing vector field on the spacetime manifold, $\xi = \xi^\mu \partial_\mu \neq 0$

$\Rightarrow$ The integral lines of the fundamental vector field are the only admissible worldlines and they are vertical: all observers are at rest.

Timelike metric structure

Fundamental vector field: Nowhere vanishing vector field on the spacetime manifold, $\xi = \xi^\mu \partial_\mu \neq 0$

$\Rightarrow$ The integral lines of the fundamental vector field are the only admissible worldlines and they are vertical: all observers are at rest. This is the origin of the nickname "Carroll" (Lévy-Leblond, 1965).



Timelike metric structure

Fundamental vector field: Nowhere vanishing vector field on the spacetime manifold, $\xi = \xi^\mu \partial_\mu \neq 0$

An affine parameter u of this congruence of Carroll worldlines (i.e. $\xi = \partial/\partial u$) is a Carroll time.

Spacelike metric structure

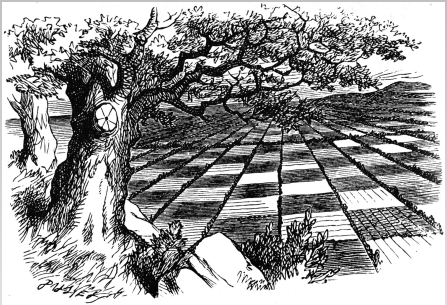
Carrollian metric: Positive semi-definite metric γ on the spacetime $\mathscr{M}$ whose kernel is spanned by the fundamental vector field

$$\begin{cases} \gamma_{\mu\nu} V^\mu W^\nu \geqslant 0 \\ \gamma_{\mu\nu} V^\mu = 0 \quad \Leftrightarrow \quad V^\mu = f\,\xi^\mu \end{cases}$$

Remark: There is a one-to-one correspondence between *invariant* Carrollian metrics γ on $\mathscr{M}$ and Riemannian metrics $\bar{\gamma}$ on the base $\bar{\mathscr{M}}$.

Spacelike metric structure

An invariant Carrollian metric allows Alice to measure distances and angles on the base manifold $\bar{\mathscr{M}}$.



Carrollian structure

(Invariant) Carrollian structure: following two data

1. Fundamental vector field
2. (Invariant) Carrollian metric

Example : Future null infinity $\mathscr{I}^+$ (or Past lightcone $\mathscr{N}^-$)

- Coordinates (u, θ^i) on $\mathscr{I}^+ \cong \mathbb{R} \times S^d$
- Null vector field $\xi = \frac{\partial}{\partial u}$
- Carrollian metric = pullback of the metric on the unit sphere

$$ds^2 = \gamma_{ij}(\theta)\, d\theta^i d\theta^j = d\ell^2_{S^d}$$

Carrollian isometries

Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Fundamental vector field $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$

Remark: For an invariant Carrollian structure, Carrollian isometries project onto isometries of the Riemannian metric $\bar{\gamma}$ on the base $\bar{\mathscr{M}}$

Carrollian isometries

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Example: Vertical automorphisms of the principal $\mathbb{R}$-bundle

$$u' = u + f(x)\,, \quad x' = x\,,$$

which are interpreted as "supertranslations" in the BMS context.

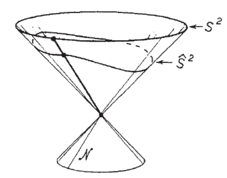

Fig. 2 in (Penrose, 1974)

Bondi-Metzner-Sachs as Conformal Carroll

Conformal Carrollian structure

Conformal Carrollian structure: equivalence class of Carrollian structures with respect to equivalence relation

1. Fundamental vector fields $\xi \sim \Omega^{-1}\xi$
2. (Invariant) Carrollian metrics $\gamma \sim \Omega^{2}\gamma$ (with $\mathcal{L}_{\xi}\Omega = 0$)

Conformal Carrollian isometries

Conformal Carrollian isometry: diffeomorphism of $\mathscr{M}$ such that

1. (Conformal scaling) $\xi' = \Omega^{-1}\xi$
2. (Conformal isometry) $\gamma' = \Omega^2\gamma$

Conformal Carrollian isometries

Conformal Carrollian isometry: diffeomorphism of $\mathscr{M}$ such that

1. (Conformal scaling) $\xi' = \Omega^{-1}\xi$
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Example: For future null infinity $\mathscr{I}^+$ at the conformal boundary of compactified Minkowski spacetime,

Theorem ((Penrose, 1965) revisited (Duval-Gibbons-Horvathy, 2014))

BMS transformations = Conformal Carrollian isometries

Conformal Carroll-Killing vector field

Conformal Carroll-Killing vector field: $X \in \mathfrak{X}(\mathscr{M})$ such that

1. (projectable) $\mathcal{L}_X \xi = f\, \xi$
2. (conformal Killing) $\mathcal{L}_X \gamma = -2f\, \gamma$

Conformal Carroll-Killing vector field

Consider an invariant conformal Carrollian structure.

The projection $\bar{X} = \pi_*(X)$ on the base $\bar{\mathscr{M}}$ of a conformal Carroll-Killing vector field X on $\mathscr{M}$ is a conformal Killing vector field $\bar{X}$ on $\bar{\mathscr{M}}$.

Conformal Carroll-Killing vector field: $X \in \mathfrak{X}(\mathscr{M})$ such that

1. (projectable) $\mathcal{L}_X \xi = f\,\xi$ with $\mathcal{L}_\xi f = 0$
2. (conformal Killing) $\mathcal{L}_{\bar{X}} \bar{\gamma} = -2\bar{f}\bar{\gamma}$

Conformal Carroll-Killing vector field

Lemma

The conformal Carroll-Killing vector fields on $\mathscr{I}^\pm \cong \mathbb{R} \times S^d$ span the (extended) BMS algebra

$$(\mathfrak{e})\mathfrak{bms}_{d+2} = C^\infty(S^d) \ni \mathfrak{conf}(S^d)$$

where the elements of $C^\infty(S^d)$ transform as densities of conformal weight -1 under $\mathfrak{conf}(S^d)$.

Higher-spin extension

Towards higher-spin extension

First proposal

Main idea: rephrase the Carroll-Killing equations as commutation relations

Towards higher-spin extension

Consider an invariant conformal Carrollian structure.

Conformal Carroll-Killing vector field: $X \in \mathfrak{X}(\mathscr{M})$ such that

1. (projectable) $\mathcal{L}_X \xi = f\,\xi \quad \Leftrightarrow \quad [X, \xi\,] = f\,\xi$
2. (conformal Killing) $\mathcal{L}_{\bar{X}} \bar{\gamma} = -2\bar{f}\bar{\gamma} \quad \Leftrightarrow \quad [\bar{X}, \bar{\Delta}\,] = 2f\,\bar{\Delta}$

where $\bar{\Delta}$ is the conformal Laplacian of the Riemannian metric $\bar{\gamma}$.

Towards higher-spin extension

Higher-spin recipe: Vector field → Differential operator

Higher-spin extension

Consider an invariant conformal Carrollian structure.

Conformal Carroll-Laplacian symmetry: $\hat{D} \in \mathcal{D}(\mathscr{M})$ such that

1. (projectable differential operator) $[\hat{D}, \hat{\xi}\,] = \hat{F} \circ \hat{\xi}$
2. (conformal Laplacian symmetry) $[\hat{\hat{D}}, \hat{\hat{\Delta}}\,] = 2\hat{\hat{F}} \circ \hat{\hat{\Delta}}$

for some invariant differential operator $\hat{F}$ on $\mathscr{M}$, i.e. $[\hat{F}, \hat{\xi}\,] = 0$.

Higher-spin extension

Consider an invariant conformal Carrollian structure.

Lemma

- *The conformal Carroll-Laplacian symmetries span an associative algebra* $\mathtt{hsbms}(\mathscr{M})$, *i.e. the sum and the product of two such symmetries is still a symmetry.*
- *The algebra* $\mathcal{D}_{vsym}(\mathscr{M})$ *of vertical conformal Carroll-Laplacian symmetries spans an ideal of the algebra* $\mathtt{hsbms}(\mathscr{M})$.

$$\hat{D} \in \mathcal{D}_{vsym}(\mathscr{M}) \quad \Leftrightarrow$$

$$\hat{D} \in \mathtt{hsbms}(\mathscr{M}) \quad \text{and} \quad \hat{D} = \hat{F} \circ \hat{\xi}\,,$$

Remark: The vertical invariant differential operators are natural higher-spin extensions of infinitesimal supertranslations.

Higher-spin extension

Theorem

$$\texttt{hsbms}(\mathscr{I}^+) \cong \mathcal{D}_{vsym}(\mathscr{I}^+) \rtimes \texttt{hs}(S^d)$$

where

- $\texttt{hsbms}(\mathscr{I}^+)$ = *associative algebra of conformal Carroll-Laplacian symmetries on* $\mathscr{I}^+ \cong \mathbb{R} \times S^d$
- $\mathcal{D}_{vsym}(\mathscr{I}^+)$ = *ideal of vertical conformal Carroll-Laplacian symmetries on* $\mathscr{I}^+ \cong \mathbb{R} \times S^d$
- $\texttt{hs}(S^d)$ = *Eastwood-Vasiliev off-shell higher-spin algebra of symmetries of the conformal Laplacian on* S^d

Remark: The lift of symmetries of the conformal Laplacians are the higher-spin extension of infinitesimal (super)rotations.

Higher-spin extension

Theorem

$$\mathtt{hsbms}(\mathscr{I}^+) \cong \mathcal{D}_{\mathsf{vsym}}(\mathscr{I}^+) \ltimes \mathtt{hs}(S^d)$$

where

- $\mathtt{hsbms}(\mathscr{I}^+)$ = *associative algebra of conformal Carroll-Laplacian symmetries on* $\mathscr{I}^+ \cong \mathbb{R} \times S^d$
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Idea of the proof: The symmetries of conformal Carroll-Laplacians are projectable differential operators. The kernel of the pushforward is $\mathcal{D}_{\mathsf{vsym}}(\mathscr{I}^+)$ and the image is $\mathtt{hs}(S^d)$ by definition.

On-shell version

Definition (Intrinsic)

Wick-rotated Rac: module spanned by $\phi \in C^\infty(\mathscr{I}^+)$ such that

1. (invariant) $\mathcal{L}_\xi \phi = 0$
2. (conformal Laplace) $\hat{\bar{\Delta}}\bar{\phi} = 0$

Remarks:

- Faithful irreducible module of $\mathfrak{so}(d+1,1)$
- Unfaithful module of $(\mathfrak{e})\mathfrak{bms}_{d+2}$ and $\mathfrak{iso}(d+1,1)$

On-shell version

Definition (Holographic)

Wick-rotated Rac: module spanned by boundary data of solutions to d'Alembert equation with homogeneity degree $1-\frac{d}{2}$

$$\begin{cases} \Box\,\Phi(r,u,\theta)=0 \\ \phi(u,\theta)=\lim\limits_{r\to\infty}\left[\,r^{\frac{d}{2}-1}\Phi(r,u,\theta)\,\right] \end{cases}$$

where (r,u,θ^i) are Bondi coordinates on Minkowski spacetime $\mathbb{R}^{d+1,1}$.

On-shell version

Trivial Conformal Carroll-Laplacian symmetry: symmetry $\hat{D}$ acting trivially on Wick-rotated Rac

On-shell version

Trivial Conformal Carroll-Laplacian symmetry:
$\hat{D}$ acting trivially on Wick-rotated Rac

Theorem (Eastwood, 2002)

The algebra of non-trivial conformal Carroll-Laplacian symmetries on $\mathscr{I}^+ \cong \mathbb{R} \times S^d$ is isomorphic to the Eastwood-Vasiliev on-shell higher-spin algebra of symmetries of the conformal Laplacian on S^d.

Towards higher-spin extension

Second proposal

Main idea: consider Sachs representation instead

Sachs representation

Definition (Intrinsic)

Sachs: unitary module spanned by square-integrable densities $\psi \in C^\infty(\mathscr{M})$ of conformal weight $d/2$ and positive Carrollian energy endowed with the Hermitian product

$$\langle\!\langle \psi_1 \,|\, \psi_2 \rangle\!\rangle = i \int_{\mathscr{M}} \psi_1^* \, d\psi_2 \wedge \mathcal{V}$$

where $\mathcal{V} = \pi^*(\bar{\star}1)$ is the pullback of the volume form $\bar{\star}1$ of the base manifold $\bar{\mathscr{M}}$.

Local expression: Sachs (1962)

$$\langle\!\langle \psi_1 \,|\, \psi_2 \rangle\!\rangle = i \int du\, d^dx \sqrt{\gamma}\, \psi_1^* \frac{\partial \psi_2}{\partial u} = \langle \psi_1 \,|\, \hat{H} \,|\, \psi_2 \rangle$$

Sachs representation

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where $\mathcal{V} = \pi^*(\bar{\star}1)$ is the pullback of the volume form $\bar{\star}1$ of the base manifold $\bar{\mathscr{M}}$.

Carrollian Physics interpretation:

Norm-squared $\quad \langle\!\langle \psi \,|\, \psi \rangle\!\rangle = \langle \hat{H} \rangle_\psi \quad$ Mean value of Carroll Hamiltonian

Sachs representation

Definition (Holographic)

Sachs: module spanned by boundary data of radiation solutions to d'Alembert equation

$$\begin{cases} \Box\, \Phi(r,u,\theta) = 0 \\ \psi(u,\theta) = \lim\limits_{r\to\infty} \left[\, r^{\frac{d}{2}}\, \Phi(r,u,\theta) \,\right] \end{cases}$$

where (r, u, θ^i) are Bondi coordinates on Minkowski spacetime $\mathbb{R}^{d+1,1}$.

Higher-spin extension

Higher-spin recipe:

higher-spin algebra ≡ universal enveloping algebra / annihilator

$$\mathcal{U}\Big((\mathfrak{e})\mathfrak{bms}_{d+2}\Big)\,/\,\mathrm{Ann}(\mathrm{Sachs})$$

Conclusion

Summary of results

- Identification of
  - Wick-rotated Rac
  - Sachs

  as possible analogues (in Minkowski spacetime)
  to the scalar singleton (in Anti de Sitter spacetime)
- Geometric and manifestly BMS-invariant formulation of Sachs Hermitian product
- Definition of possible higher-spin extensions of (extended) BMS algebra which
  - contain the higher-spin extension of Poincaré algebra
  - are symmetry algebras of the corresponding BMS modules

Thank you for your attention

illustrations of Alice from
John Tenniel (1820-1914)