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SPIN(7) INSTANTONS IN EIGHT DIMENSIONS

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based on
A.S., arXiv: 2102.07415[hep-th]

BPST instantons

• Consider $A(x) = A_\mu(x) dx_\mu$ in $\mathbb{R}^4$.

• Require that $F = dA - iA \wedge A$ decays fast enough at $x \to \infty$.

Then

$$q \;=\; \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\{F \wedge F\} \;=\; \text{integer}$$

Indeed:

$$\mathrm{Tr}\{F \wedge F\} = d\omega_3, \qquad \omega_3 = \mathrm{Tr}\left\{A \wedge F + \frac{i}{3} A \wedge A \wedge A\right\}.$$

Hence

$$q \;=\; \frac{1}{8\pi^2}\int_{S^3} \omega_3 \,.$$

At infinity, $A = i g^{-1} dg$. Hence

$$q \;=\; \frac{1}{24\pi^2}\int_{S^3} \mathrm{Tr}\{g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\}.$$

- Nontrivial mappings $S^3 \to SU(2)$.

$$\pi_3[SU(2)] \ = \ \mathbb{Z}\,.$$

The simplest nontrivial mapping

$$g \ = \ \frac{\mathbf{1}x_4 - ix_m\sigma_m}{r}\,, \tag{1}$$

- All the gauge fields with the asymptotics

$$A \ = \ ig^{-1}dg \ = \ \frac{\Sigma_{\mu\nu}x_\nu dx_\mu}{r^2}\,,$$

where

$$\Sigma_{\mu\nu} \ = \ \frac{i}{2}(\sigma_\mu\sigma_\nu^\dagger - \sigma_\nu\sigma_\mu^\dagger)$$

with $\sigma_\mu = (\vec{\sigma}, i)$ and $\sigma_\mu^\dagger = (\vec{\sigma}, -i)$,
have the Pontryagin index $q = 1$.

- The variable change

$$\alpha_m \;=\; -\frac{x_m}{\|\vec{x}\|} \arccos\left(\frac{x_4}{r}\right) .$$

gives the exponential parameterization:

$$g \;=\; \exp\{i\alpha_m\sigma_m\} = \mathbf{1}\cos\alpha + \frac{i\sin\alpha}{\alpha}\alpha_m\sigma_m .$$

In these terms,

$$g^{-1}\partial_m g = -iA_m \;=\;$$
$$i\left[\frac{\cos\alpha\sin\alpha}{\alpha}\sigma_m + \frac{\sin^2\alpha}{\alpha^2}\alpha_k\varepsilon_{kml}\sigma_l +\right.$$
$$\left.\frac{\alpha_m\alpha_k}{\alpha^2}\left(1 - \frac{\cos\alpha\sin\alpha}{\alpha}\right)\sigma_k\right] .$$

Instantons:

Special configurations with nonzero q realizing the minimum of the action

$$S = \int_{\mathbb{R}^4} \mathrm{Tr}\,\{F \wedge *F\} = \frac{1}{2}\int d^4x\,\mathrm{Tr}\,\{F_{\mu\nu}F_{\mu\nu}\}\,.$$

If $q = 1$, they are self-dual, $F = *F$.

The explicit expression:

$$A = \frac{\Sigma_{\mu\nu}x_\nu dx_\mu}{r^2 + \rho^2}$$

(center at the origin).

Spin(8) instantons

- Eight dimensions.
- Topological charge:

$$q = \frac{1}{384\pi^4} \int_{\mathbb{R}^8} \mathrm{Tr}\left\{F \wedge F \wedge F \wedge F\right\}.$$

- Note that

$$\mathrm{Tr}\left\{F \wedge F \wedge F \wedge F\right\} = d\omega_7 \,,$$

where

$$\omega_7 = \mathrm{Tr}\left\{ AF^3 + \frac{2i}{5} F^2 A^3 + \frac{i}{5} AFA^2F \right.$$
$$\left. -\frac{1}{5} FA^5 - \frac{i}{35} A^7 \right\}.$$

- If $F \to 0$ when $x \to \infty$,

$$q = -\frac{1}{2^7 \cdot 3 \cdot 35\,\pi^4} \int_{S^7} \mathrm{Tr}\left\{(g^{-1}dg)^7\right\}.$$

- A top. nontrivial embedding $S^7 \to Spin(8)$ with top. charge $q = 1$:

$$g = \frac{\mathbf{1}x_8 + i x_m \Gamma_m}{r}$$

with $m = 1, \ldots, 7$.

$\Gamma_{j=1,\ldots,7}$ are purely imaginary antisymmetric 8×8 matrices satisfying the Clifford algebra,

$$\Gamma_m \Gamma_n + \Gamma_n \Gamma_m = 2\delta_{mn} \mathbf{1} \,,$$

and the relation

$$\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_7 = i \mathbf{1} \,.$$

One of many explicit representations for Γ_m is

$$\begin{aligned}
\Gamma_1 &= -\sigma_2 \otimes \sigma_2 \otimes \sigma_2; \quad \Gamma_2 = \mathbf{1} \otimes \sigma_1 \otimes \sigma_2; \\
\Gamma_3 &= \mathbf{1} \otimes \sigma_3 \otimes \sigma_2; \quad \Gamma_4 = -\sigma_1 \otimes \sigma_2 \otimes \mathbf{1}; \\
\Gamma_5 &= \sigma_3 \otimes \sigma_2 \otimes \mathbf{1}; \quad \Gamma_6 = \sigma_2 \otimes \mathbf{1} \otimes \sigma_1; \\
\Gamma_7 &= \sigma_2 \otimes \mathbf{1} \otimes \sigma_3 \,. \qquad (2)
\end{aligned}$$

- Generators of $Spin(8)$:

$$\Sigma_{MN} = \frac{i}{2} (\Gamma_M \Gamma_N^\dagger - \Gamma_N \Gamma_M^\dagger) \,,$$

where

$$\Gamma_M = (\vec{\Gamma}, i), \qquad \Gamma_M^\dagger = (\vec{\Gamma}, -i) \,,$$

- Σ_{mn} are the generators of $Spin(7)$.

  - Exponential parameterization:

$$g = \exp\{i \alpha_j \Gamma_j\} = \mathbf{1} \cos\alpha + i \frac{\sin\alpha}{\alpha} \alpha_j \Gamma_j \,,$$

Instanton

is a configuration of topological charge $q = 1$ that has the asymptotics

$$A = ig^{-1}dg = \frac{\Sigma_{NM} x_N dx_M}{r^2}$$

and realizes the minimum of the functional

$$\int_{R^8} \mathrm{Tr}\,\{F \wedge F \wedge \star(F \wedge F)\}$$

It satisfies the *nonlinear* self-duality condition

$$F \wedge F = \star(F \wedge F)$$

The explicit form for the instanton centered at the origin:

$$A = ig^{-1}dg = \frac{\Sigma_{NM} x_N dx_M}{r^2 + \rho^2}.$$

[Grossman, Kephart, Stasheff, Commun. Math. Phys. **96** (1984) 431; Tchrakian, Phys. Lett. **B150** (1985) 360.]

Antiinstanton:

$$A = \frac{\tilde{\Sigma}_{NM} x_N dx_M}{r^2 + \rho^2}$$

with

$$\tilde{\Sigma}_{MN} = \frac{i}{2}(\Gamma^{\dagger}_{M}\Gamma_{N} - \Gamma^{\dagger}_{N}\Gamma_{M}).$$

It has $q = -1$.

• Multiinstanton configurations (generalizing AHDM 4d constructions) — see [Nakamura, Sasaki, Takesue, Nucl. Phys. **B214** (1983) 452.]

• All these instantons are characterized by a *single* topological charge q. But the relevant homotopy group is

$$\pi_7[Spin(8)] = \mathbb{Z} \times \mathbb{Z}.$$

implying the existence of *two* top. invariants.

Generic top. nontrivial mappings $S^7 \to Spin(8)$?

The mappings $S^7 \to Spin(7)$ corresponding to $\pi_7[Spin(7)] = \mathbb{Z}$?

The corresponding instantons?

Octonionic instanton

[Corrigan, Devchand, Fairlie, Nuyts, Nucl. Phys. **B214** (1983) 452; S. Fubini and H. Nicolai, Phys. Lett. **B155** (1985) 369.]

• A different story. It satisfies certain linear quasi-self-duality conditions:

$$F_{MN} = \frac{1}{2} C_{MNPQ} F_{PQ}.$$

• The embeddings $S^7 \to Spin(8)$ discussed above were associated with the fiber bundle

$$Spin(8) \stackrel{Spin(7)}{\longrightarrow} S^7 \,.$$

• The homotopy $\pi_7[Spin(7)] = \mathbb{Z}$ is associated with the fiber bundle

$$Spin(7) \stackrel{G_2}{\longrightarrow} S^7 \,.$$

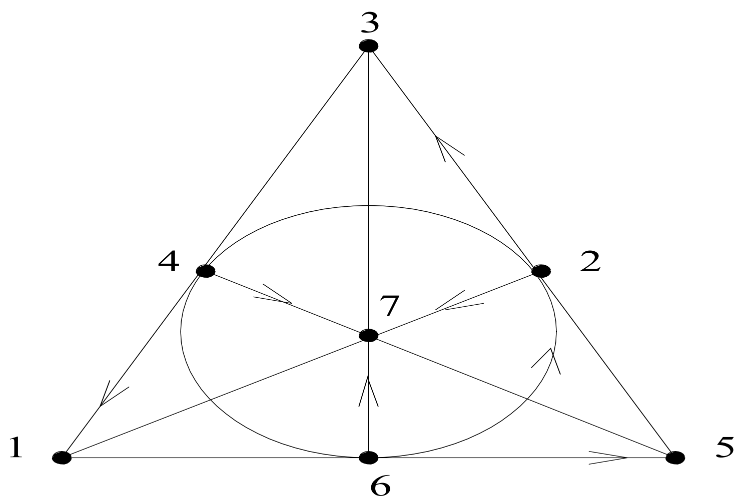
Spin(7) constructions.

Consider the expression

$$g = \exp\left\{\frac{1}{2}\alpha_m f_{mnk}\Gamma_n\Gamma_k\right\}, \qquad (3)$$

where f_{mnp} is antisymmetric and

$f_{165} = f_{134} = f_{127} = f_{235} = f_{246} = f_{367} = f_{475} = 1.$



- $[\Gamma_m, \Gamma_n] \in spin(7)$; $g \in Spin(7)$.

THEOREM: The expression (3) considered in the range $0 \leq \|\vec{\alpha}\| \leq \pi$ *represents a map* $S^7 \longrightarrow Spin(7)$ *associated with the fiber bundle* $Spin(7) \xrightarrow{G_2} S^7$.

Proof. To prove that $g(\vec{\alpha})$ represents a map $S^7 \to Spin(7)$, it suffices to show that $g(|\vec{\alpha}| = \pi) = -\mathbf{1}$.

• Let first $\alpha_m = \pi\delta_{m1}$. Then

$$g = \exp\{\pi(\Gamma_3\Gamma_4 - \Gamma_5\Gamma_6 + \Gamma_2\Gamma_7)\} =$$
$$\exp\{\pi\Gamma_3\Gamma_4\}\exp\{-\pi\Gamma_5\Gamma_6\}\exp\{\pi\Gamma_2\Gamma_7\}$$
$$= (-\mathbf{1})^3 = -\mathbf{1}.$$

• We now rotate $\vec{\alpha}$. It does not follow immediately that $g(|\vec{\alpha}| = \pi) = -\mathbf{1}$ also after rotation: f_{mnk} is not an $SO(7)$ - invariant. Still it is true: Rotate $\vec{\alpha}$ in the plane $\{12\}$:

$$\alpha_1 = \pi\cos\phi, \quad \alpha_2 = \pi\sin\phi, \quad \alpha_{3,\ldots,7} = 0.$$

One can then show that

$$\ln g \; = \; \pi\Gamma_3\Gamma_4' - \pi\Gamma_5'\Gamma_6 + \pi\Gamma_2'\Gamma_7, \tag{4}$$

where

$\Gamma_4' = \Gamma_4\cos\phi + \Gamma_5\sin\phi$, $\Gamma_5' = \Gamma_5\cos\phi - \Gamma_4\sin\phi$, $\Gamma_2' = \Gamma_2\cos\phi - \Gamma_1\sin\phi$.

We may observe that the matrices $\Gamma_3, \Gamma_6, \Gamma_7$, $\Gamma'_4, \Gamma'_5, \Gamma'_2$ and $\Gamma'_1 = \Gamma_1 \cos\phi + \Gamma_2 \sin\phi$ still obey the Clifford algebra, which means that the three terms in (4) still commute, the exponential can be "disentangled" and we still have $g = (-\mathbf{1})^3 = -\mathbf{1}$.

The same is true for rotations in other planes.

• Any $\vec{\alpha}$ of norm π can be "reached" from $\alpha_m = \pi\delta_{m1}$ by a set of such elementary rotations, and we conclude that, for such $\vec{\alpha}$, $g(\vec{\alpha}) = -\mathbf{1}$, indeed.

• To prove that this mapping is associated with the fiber bundle $Spin(7) \xrightarrow{G_2} S^7$, we need to prove that the set of all $g(\vec{\alpha})$ can be considered as a base in this fiber bundle. And this is true, because $\ln g$ is not in the subalgebra $g_2 \subset spin(7)$, but is orthogonal to it:

$$[f_{mnk}\Gamma_n\Gamma_k, h] \;=\; 0, \qquad \text{if } h \in g_2 \,.$$

□

Exponential expanded:

$$g = \cos^3\alpha\,\mathbf{1} - i\cos^2\alpha\sin\alpha\,\frac{\alpha_m T_m}{\alpha} +$$
$$\cos\alpha\sin^2\alpha\,\frac{\alpha_m\alpha_n\Gamma_m T_n}{\alpha^2} - i\sin^3\alpha\,\frac{\alpha_m\Gamma_m}{\alpha}. \qquad (5)$$

with

$$T_m = \frac{i}{2} f_{mnk}\Gamma_n\Gamma_k.$$

In Cartesian coordinates:

$$g = \frac{x_8^3 - ix_8^2\, x_m T_m + x_8\, x_m\Gamma_m\, x_n T_n - ix_m x_m\, x_n\Gamma_n}{r^3}.$$

• The expressions for $A_M = ig^{-1}\partial_M g$ are bulky and ugly.

• The integral

$$q = -\frac{1}{2^7\cdot 3\cdot 35\,\pi^4}\int_{S^7}\mathrm{Tr}\,\{(g^{-1}dg)^7\}\,.$$

for the topological charge was calculated numerically:

$$q = .91 \pm .09$$

(40 min. of Mathematica running on my laptop).

Other groups and other dimensions

• Homotopy $\pi_7[Spin(6)] = \mathbb{Z}$ associated with the fiber bundle $Spin(6) \stackrel{SU(3)}{\longrightarrow} S^7$.

• Homotopy $\pi_7[Spin(5)] = \mathbb{Z}$ associated with the fiber bundle $Spin(5) \stackrel{SU(2)}{\longrightarrow} S^7$.

• No known nice expressions for the mappings. But the mappings exist and the instantons realizing the minimum of

$$\int_{\mathbb{R}^8} \mathrm{Tr}\,\{F \wedge F \wedge \star(F \wedge F)\}$$

exist too.

Dimension $4r$

- $\pi_{4r-1}[Sp(r)] \;=\; \mathbb{Z}.$

- No nice formula again. The instantons realize the minimum of the functional

$$\int_{\mathbb{R}^{4r}} \mathrm{Tr}\,\{\overbrace{F\wedge\ldots\wedge F}^{r}\;\star(\overbrace{F\wedge\ldots\wedge F}^{r})\}$$

and satisfy the relations

$$\overbrace{F\wedge\ldots\wedge F}^{r} \;=\; \pm\star\left(\overbrace{F\wedge\ldots\wedge F}^{r}\right).$$

Dimension 6

• The mapping $S^5 \to SU(3)$ associated with the fiber bundle $SU(3) \stackrel{SU(2)}{\longrightarrow} S^5$ is known.

The vector

$$V = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \qquad |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$$

maps for $|\alpha_1| > 0$ to

$$g_1 = \begin{pmatrix} -\frac{\alpha_1 \alpha_2^*}{\sqrt{1-|\alpha_2|^2}} & -\frac{\alpha_3^*}{\sqrt{1-|\alpha_2|^2}} & \alpha_1 \\ \sqrt{1-|\alpha_2|^2} & 0 & \alpha_2 \\ -\frac{\alpha_3 \alpha_2^*}{\sqrt{1-|\alpha_2|^2}} & \frac{\alpha_1^*}{\sqrt{1-|\alpha_2|^2}} & \alpha_3 \end{pmatrix} \in SU(3)\,.$$

[Khanna, Mukhopadhyay, Simon, Mukunda, Ann. Phys. **253** (1997) 55.]

• Similar expressions $g_{2,3}$ for the regions $|\alpha_2| > 0$ and $|\alpha_3| > 0$. S^5 is described as a union of these three regions.

• The topological charge integral

$$q \ = \ \pm\frac{1}{480\pi^3}\int_{S^5}\mathrm{Tr}\{(g^{-1}dg)^5)$$

should give 1 or -1 for this mapping

• No self-duality conditions. Instantons exist only in dimensions $D = 4r$.