Higher-dimensional invariants in $6D$ SYM theory

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Outline

Motivations and contents

$6D$ superspaces and superfields

$\mathcal{N} = (1,1)$ on-shell harmonic superspace

Invariants in $\mathcal{N} = (1,1)$ superspace: **d=6, 8, 10**

d=12 invariants

Passing to $\mathcal{N} = (1,0)$ SYM superfields

Summary and outlook

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Outline

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d=12 invariants

Passing to $\mathcal{N} = (1,0)$ SYM superfields

Summary and outlook

Outline

Motivations and contents

$6D$ superspaces and superfields

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d=12 invariants

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Maximally extended gauge theories (with 16 supersymmetries) are under intensive study for the last few years:

$$\mathcal{N} = 4,\ 4D; \qquad \mathcal{N} = (1,1),\ 6D; \qquad \mathcal{N} = (1,0),\ 10D\,.$$

- $\mathcal{N} = 4,\ 4D$ SYM is UV finite and perhaps completely integrable.
- $\mathcal{N} = (1,1),\ 6D$ SYM is not renormalizable by formal power counting (the coupling constant is dimensionful) but is also expected to possess some unique properties.
- Its amplitudes respect the so-called "dual conformal symmetry" like its $4D$ counterpart.
- It provides the effective field theory description of the low energy limit of D5-brane dynamics.
- $\mathcal{N} = (1,1)$ SYM is anomaly free (Frampton, Kephart, 1983, *et al*), as distinct from $\mathcal{N} = (1,0)$ SYM.
- $\mathcal{N} = (1,1)$ and $\mathcal{N} = (1,0)$ SYM are analogs of $\mathcal{N} = 8$ supergravity (also non-renormalizable by power-counting).

- The full world-volume action of D5-brane is expected to be of non-abelian Born-Infeld type, generalizing the $\mathcal{N} = (1,1)$ SYM action (Tseytlin, 1997).
- The perturbative explicit calculations in $\mathcal{N} = (1,1)$ SYM (as a low-energy limit of type II superstrings) show a lot of unexpected cancelations of the UV divergencies.
- The theory is UV-finite up to 2 loops, while at 3 loops only a single-trace counterterm of canonical dim 10 is required. The allowed double-trace counterterms do not appear (Bern *et al*, 2010, 2011; Berkovits *et al*, 2009; Bjornsson *et al*, 2011, 2012).
- To understand these peculiarities, the maximally supersymmetric off-shell formulations are needed!
- Maximum what one can achieve off shell in $6D$ is $\mathcal{N} = (1,0)$ SUSY. The most natural off-shell formulation of $\mathcal{N} = (1,0)$ SYM - in harmonic $\mathcal{N} = (1,0), 6D$ superspace (Howe *et al*, 1985; Zupnik, 1986) as a generalization of $\mathcal{N} = 2, 4D$ HSS (Galperin *et al*, 1984).

- $[\mathcal{N}=(1,0)$ SYM + $6D$ hypermultiplet] = $[\mathcal{N}=(1,1)$ SYM], with the second hidden on-shell $\mathcal{N}=(0,1)$ SUSY.

- How to construct higher-dimension $\mathcal{N}=(1,1)$ invariants?

- In Bossard, Ivanov, Smilga, JHEP 1512 (2015) 085 there was developed a new approach to constructing higher-dimension $\mathcal{N}=(1,1)$ invariants based on the concept of on-shell $\mathcal{N}=(1,1)$ harmonic superspace (Bossard, Howe & Stelle, 2009).

- The hidden supersymmetry tells us nothing about the precise coefficients before invariants. To determine them, one should develop the $\mathcal{N}=(1,0)$ superfield perturbation theory. This has been done in (Buchbinder, Ivanov, Merzlikin, Stepanyantz, 1609.00975, 1612.03190, 1704.02530).

$6D$ superspaces

- Standard $\mathcal{N}=(1,0), 6D$ superspace (Howe, Sierra & Townsend, 1983):

$$z=(x^M,\theta^a_i)\,,\quad M=0,\ldots,5\,,\; a=1,\ldots,4\,,\; i=1,2\,,$$

with Grassmann pseudoreal θ^a_i.

- Harmonic $\mathcal{N}=(1,0), 6D$ superspace (Howe, Stelle & West, 1985; Zupnik, 1986):

$$Z:=(z,u)=(x^M,\theta^a_i,u^{\pm i})\,,\quad u^-_i=(u^+_i)^*, u^{+i}u^-_i=1\,,\; u^{\pm i}\in SU(2)_R/U(1)\,.$$

- *Analytic* $\mathcal{N}=(1,0), 6D$ superspace:

$$\zeta:=(x^M_{(\mathrm{an})},\theta^{+a},u^{\pm i})\subset Z\,,\quad x^M_{(\mathrm{an})}=x^M+\frac{i}{2}\theta^a_k\gamma^M_{ab}\theta^b_l u^{+k}u^{-l},\quad \theta^{\pm a}=\theta^a_i u^{\pm i}\,.$$

- Basic differential operators in the analytic basis:

$$D^+_a=\partial_{-a}\,,\quad D^-_a=-\partial_{+a}-2i\theta^{-b}\partial_{ab}\,,$$
$$D^0=u^{+i}\frac{\partial}{\partial u^{+i}}-u^{-i}\frac{\partial}{\partial u^{-i}}+\theta^{+a}\partial_{+a}-\theta^{-a}\partial_{-a}$$
$$D^{++}=\partial^{++}+i\theta^{+a}\theta^{+b}\partial_{ab}+\theta^{+a}\partial_{-a}\,,\; D^{--}=\partial^{--}+i\theta^{-a}\theta^{-b}\partial_{ab}+\theta^{-a}\partial_{+a}\,,$$

where $\partial_{\pm a}\theta^{\pm b}=\delta^b_a$ and $\partial^{++}=u^{+i}\frac{\partial}{\partial u^{-i}}\,,\quad \partial^{--}=u^{-i}\frac{\partial}{\partial u^{+i}}$.

Basic superfields

- Analytic gauge $\mathcal{N} = (1,0)$ SYM connection:

$$\nabla^{++} = D^{++} + V^{++}\,, \quad \delta V^{++} = \nabla^{++}\Lambda\,, \quad \Lambda = \Lambda(\zeta)\,.$$

- Second harmonic (non-analytic) connection:

$$\nabla^{--} = D^{--} + V^{--}\,, \quad \delta V^{--} = \nabla^{--}\Lambda\,.$$

- Related by the harmonic flatness condition

$$[\nabla^{++}, \nabla^{--}] = D^0 \;\Rightarrow\; D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0$$
$$\Rightarrow\; V^{--} = V^{--}(V^{++}, u^{\pm})\,.$$

- Wess-Zumino gauge:

$$V^{++} \;=\; i\theta^{+a}\theta^{+b}A_{ab} + 2(\theta^+)^3_a\lambda^{-a} - 3(\theta^+)^4\mathcal{D}^{--}\,.$$

Here A_{ab} is the gauge field, $\lambda^{-a} = \lambda^{ai}u^-_i$ is the gaugino and $\mathcal{D}^{--} = \mathcal{D}^{ik}u^-_i u^-_k$, where $\mathcal{D}^{ik} = \mathcal{D}^{ki}$, are the auxiliary fields.

- Covariant derivatives

$$\nabla^-_a := [\nabla^{--}, D^+_a] = D^-_a + \mathcal{A}^-_a\,, \qquad \nabla_{ab} = \frac{1}{2i}[D^+_a, \nabla^-_b] = \partial_{ab} + \mathcal{A}_{ab}\,,$$
$$\mathcal{A}^-_a(V) = -D^+_a V^{--}\,, \quad \mathcal{A}_{ab}(V) = \frac{i}{2} D^+_a D^+_b V^{--}\,,$$
$$[\nabla^{++}, \nabla^-_a] = D^+_a\,, \quad [\nabla^{++}, D^+_a] = [\nabla^{--}, \nabla^-_a] = [\nabla^{\pm\pm}, \nabla_{ab}] = 0\,.$$

- Covariant off-shell superfield strengths

$$[D^+_a, \nabla_{bc}] = \frac{i}{2}\varepsilon_{abcd} W^{+d}\,, \qquad [\nabla^-_a, \nabla_{bc}] = \frac{i}{2}\varepsilon_{abcd} W^{-d}\,,$$
$$W^{+a} = -\frac{1}{6}\varepsilon^{abcd} D^+_b D^+_c D^+_d V^{--}\,, \quad W^{-a} := \nabla^{--} W^{+a}\,,$$
$$\nabla^{++} W^{+a} = \nabla^{--} W^{-a} = 0\,, \quad \nabla^{++} W^{-a} = W^{+a}\,,$$
$$D^+_b W^{+a} = \delta^a_b F^{++}\,, \quad F^{++} = \frac{1}{4} D^+_a W^{+a} = (D^+)^4 V^{--}\,,$$
$$\nabla^{++} F^{++} = 0\,, \quad D^+_a F^{++} = 0\,.$$

- Hypermultiplet

$$q^{+A}(\zeta) = q^{iA}(x) u^+_i - \theta^{+a}\psi^A_a(x) + \text{ An infinite tail of auxiliary fields}\,,\ \mathrm{A} = 1, 2\,.$$

$\mathcal{N} = (1,0)$ superfield actions

- The $\mathcal{N} = (1,0)$ SYM action (Zupnik, 1986):

$$S^{SYM} = \frac{1}{f^2} \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr} \int d^6x\, d^8\theta\, du_1 \ldots du_n \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+ u_2^+)\ldots(u_n^+ u_1^+)}\,,$$

$$\delta S^{SYM} = 0 \;\Rightarrow\; F^{++} = 0\,.$$

- The hypermultiplet action

$$S^q = \frac{1}{2f^2} \mathrm{Tr} \int d\zeta^{(-4)} q^{+A} \nabla^{++} q^+_A\,, \quad \nabla^{++} q^+_A = D^{++} q^+_A + [V^{++}, q^+_A]\,,$$

$$\delta S^q = 0 \;\Rightarrow\; \nabla^{++} q^{+A} = 0\,.$$

- $\mathcal{N} = (1,0)$ superfield form of $\mathcal{N} = (1,1)$ SYM action:

$$S^{(V+q)} = S^{SYM} + S^q = \frac{1}{f^2}\left(\int dZ \mathcal{L}^{\mathrm{SYM}} + \frac{1}{2}\mathrm{Tr}\int d\zeta^{(-4)} q^{+A} \nabla^{++} q^+_A\right),$$

$$\delta S^{(V+q)} = 0 \;\Rightarrow\; F^{++} + \frac{1}{2}[q^{+A}, q^+_A] = 0\,, \quad \nabla^{++} q^{+A} = 0\,.$$

It is invariant under the second $\mathcal{N} = (0,1)$ supersymmetry:

$$\delta V^{++} = \epsilon^{+A} q^+_A\,, \quad \delta q^{+A} = -(D^+)^4(\epsilon^{-A} V^{--})\,, \quad \epsilon^{\pm}_A = \epsilon_{aA}\theta^{\pm a}\,.$$

$\mathcal{N} = (1,1)$ on-shell harmonic superspace

- Extend $\mathcal{N} = (1,0)$ superspace to $\mathcal{N} = (1,1)$ one,

$$z = (x^{ab}, \theta^a_i) \;\Rightarrow\; \hat{z} = (x^{ab}, \theta^a_i, \hat{\theta}_{Aa}).$$

- Double set of covariant spinor derivatives appears,

$$\nabla^i_a = \frac{\partial}{\partial \theta^a_i} - i\theta^{bi}\partial_{ab} + \mathcal{A}^i_a \quad \hat{\nabla}^{aA} = \frac{\partial}{\partial \hat{\theta}_{Aa}} - i\hat{\theta}^A_b \partial^{ab} + \hat{\mathcal{A}}^{aA}.$$

- The defining constraints of $\mathcal{N} = (1,1)$ SYM read (Howe, Sierra & Townsend, 1983; Howe & Stelle, 1984):

$$\{\nabla^{(i}_a, \nabla^{j)}_b\} = \{\hat{\nabla}^{a(A}, \hat{\nabla}^{bB)}\} = 0\,, \quad \{\nabla^i_a, \hat{\nabla}^{bA}\} = \delta^b_a \phi^{iA}$$
$$\Rightarrow \quad \nabla^{(i}_a \phi^{j)A} = \hat{\nabla}^{a(A}\phi^{B)i} = 0 \quad \text{(By Bianchis)}\,.$$

- Next, define $\mathcal{N}=(1,1)$ HSS with the double set of $SU(2)$ harmonics (Bossard, Howe & Stelle, 2009):

$$Z=(x^{ab},\theta^a_i,u^\pm_k)\ \Rightarrow\ \hat{Z}=(x^{ab},\theta^a_i,\hat{\theta}_{Ab},u^\pm_k,u^{\hat{\pm}}_A)$$

- Then pass to the analytic basis and choose the "hatted" spinor derivatives short, $\nabla^{\hat{+}a}=D^{\hat{+}a}=\frac{\partial}{\partial\theta^{\hat{-}}_a}$. The $\mathcal{N}=(1,1)$ SYM constraints are rewritten in $\mathcal{N}=(1,1)$ HSS as

$$\{\nabla^+_a,\nabla^+_b\}=0\,,\quad \{D^{\hat{+}a},D^{\hat{+}b}\}=0\,,\quad \{\nabla^+_a,D^{\hat{+}b}\}=\delta^b_a\phi^{+\hat{+}}\,,$$
$$[\nabla^{\hat{+}\hat{+}},\nabla^+_a]=0\,,\ [\tilde{\nabla}^{++},\nabla^+_a]=0\,,\ [\nabla^{\hat{+}\hat{+}},D^{a\hat{+}}]=0\,,\ [\tilde{\nabla}^{++},D^{a\hat{+}}]=0\,,$$
$$[\tilde{\nabla}^{++},\nabla^{\hat{+}\hat{+}}]=0\,.$$

- Here

$$\nabla^+_a=D^+_a+\mathcal{A}^+_a(\hat{Z})\,,\ \tilde{\nabla}^{++}=D^{++}+\tilde{V}^{++}(\hat{\zeta})\,,\quad \nabla^{\hat{+}\hat{+}}=D^{\hat{+}\hat{+}}+V^{\hat{+}\hat{+}}(\hat{\zeta})\,,$$
$$\hat{\zeta}=(x^{ab}_{\rm an},\theta^{\pm a},\theta^{\hat{+}}_c,u^\pm_i,u^{\hat{\pm}}_A)\,,\ \text{“}3/4\ \ analytic''.$$

Solving $\mathcal{N} = (1, 1)$ SYM constraints

- The starting point is to fix, using the $\Lambda(\hat{\zeta})$ gauge freedom, the WZ gauge for the second harmonic connection $V^{\hat{+}\hat{+}}(\hat{\zeta})$

$$V^{\hat{+}\hat{+}} = i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\hat{\mathcal{A}}^{ab} + \varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\varphi_d^A u_A^{\hat{-}} + \varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}\mathcal{D}^{AB}u_A^{\hat{-}}u_B^{\hat{-}} ,$$

  with $\hat{\mathcal{A}}^{ab}, \varphi_d^A$ and $\mathcal{D}^{(AB)}$ being some $\mathcal{N} = (1, 0)$ harmonic superfields.

- Then the above constraints are reduced to some harmonic equations which can be explicitly solved.
- We have obtained that the first harmonic connection V^{++} coincides precisely with the standard $\mathcal{N} = (1, 0)$ one, $V^{++} = V^{++}(\zeta)$, while the dependence of all other geometric $\mathcal{N} = (1, 1)$ objects on the variables with "hat" proves to be strictly fixed

$$V^{\hat{+}\hat{+}} = i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\mathcal{A}^{ab} - \frac{1}{3}\epsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}D_d^+ q^{-\hat{-}} + \frac{1}{8}\epsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, q^{-\hat{-}}]$$

$$\phi^{+\hat{+}} = q^{+\hat{+}} - \theta_a^{\hat{+}}W^{+a} - i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\nabla^{ab}q^{+\hat{-}} + \frac{1}{6}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}[D_d^+ q^{-\hat{-}}, q^{+\hat{-}}]$$

$$+ \frac{1}{24}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, [q^{+\hat{-}}, q^{-\hat{-}}]] .$$

- Here, $q^{+\hat{\pm}} = q^{+A}(\zeta)u_A^{\hat{\pm}}\,,\ q^{-\hat{\pm}} = q^{-A}(\zeta)u_A^{\hat{\pm}}$ and $W^{+a}, q^{\pm A}$ are just the $\mathcal{N} = (1,0)$ superfields used previously. In the process of solving the constraints, there appeared the analyticity conditions for q^{+A}, as well as the full set of the superfield equations of motion

$$\nabla^{++}q^{+A} = 0\,, \quad F^{++} = \frac{1}{4}D_a^{+}W^{+a} = -\frac{1}{2}[q^{+A}, q_A^{+}]\,.$$

Invariants in $\mathcal{N} = (1,1)$ superspace: **d=6, 8, 10**

- The advantage of using the constrained $\mathcal{N} = (1,1)$ strengths $\phi^{+\hat{+}}$ and their covariant derivatives for constructing various invariants is their extremely simple transformation rules under $\mathcal{N} = (0,1)$ supersymmetry

$$\delta\Phi(\hat{Z}) = -\Big(\epsilon_a^{\hat{+}}\frac{\partial}{\partial\theta_a^{\hat{+}}} + \epsilon_a^{\hat{-}}\frac{\partial}{\partial\theta_a^{\hat{-}}} + 2i\epsilon_a^{\hat{-}}\theta_b^{\hat{+}}\partial^{ab}\Big)\Phi(\hat{Z}) + [\Lambda^{(compen)}, \Phi(\hat{Z})]\,,$$

  where $\Lambda^{(compen)}$ is a compensating gauge parameter.

- Lagrangian densities for all invariants can be constructed as Tr of various products of ϕ^{++} and its covariant harmonic, spinor and space-time derivatives. These are in adjoint of gauge group, so their products under the trace transform as the ordinary $\mathcal{N} = (1,1)$ superfields. Integrating them over the complete bi-HSS or its invariant subspaces, we obtain $\mathcal{N} = (1,1)$ invariant quantities.

- These super(sub)spaces are total bi-HSS (16 fermionic coordinates) and its various analytic subspaces (8 and 12 fermionic coordinates):

$$\hat{Z} = (x^{ab}, \theta^{\pm c}, \theta_d^{\hat{\pm}}, u_k^{\pm}, u_A^{\hat{\pm}}), \quad \hat{\zeta} = (x^{ab}, \theta^{+c}, \theta_d^{\hat{+}}\, u_k^{\pm}, u_A^{\hat{\pm}})$$

$$\hat{\zeta}_I = (\hat{\zeta}, \theta^{-a}), \quad \hat{\zeta}_{II} = (\hat{\zeta}, \theta_a^{\hat{-}})$$

- So we have four types of the invariants

$$S := \int d\hat{Z}\, L\,, \quad \mathcal{S} := \int d\hat{\zeta}^{(-4,\hat{-}4)} \mathcal{L}^{(+4,\hat{+}4)}\,,$$
$$\mathcal{S}_I := \int d\hat{\zeta}_I^{(0,\hat{-}4)} \mathcal{L}_I^{(0,\hat{+}4)}\,, \quad \mathcal{S}_{II} := \int d\hat{\zeta}_{II}^{(-4,0)} \mathcal{L}_{II}^{(+4,0)}\,.$$

- By definition, the dimension of the given invariant is the canonical dimension (in mass units) of the component Lagrangian density (the "microscopic" action corresponds to the dimension $\mathbf{d} = \mathbf{4}$). The dimensions of various "bricks" for constructing invariants are

$$[\nabla^{\pm}_{a}] = [\nabla^{\hat{-}a}] = [D^{\hat{+}a}] = 1/2\,,\ [\nabla^{ab}] = [\phi^{+\hat{+}}] = 1\,,\ [\nabla^{\pm\pm}] = [\nabla^{\hat{\pm}\hat{\pm}}] = 0\,,$$
$$[d\hat{Z}] = 2\,, \quad [d\hat{\zeta}^{(-4,\hat{-}4)}] = -2\,,\ [d\hat{\zeta}_I^{(0,\hat{-}4)}] = [d\hat{\zeta}_{II}^{(-4,0)}] = 0\,.$$

- Firstly we recall invariants constructed in G. Bossard, E.I., A.Smilga, 2015. The first non-trivial invariant corresponds to the canonical dimension $\mathbf{d} = \mathbf{6}$, i.e. it is dimensionless. Correspondingly, the dimensions of superfield Lagrangians defined above are

$$[L] = -4\,, \quad [\mathcal{L}^{(+4,\hat{+}4)}] = 0\,, \quad [\mathcal{L}_I^{(0,\hat{+}4)}] = [\mathcal{L}_{II}^{(+4,0)}] = -2\,.$$

All invariants except the second one are ruled out on the dimensionality grounds.

- No analytic bi-harmonic densities of the dimension 2 and the charges $(+4,\hat{+}4)$ can be constructed from the basic “bricks”. So there exist no on-shell $\mathcal{N}=(1,1)$ invariants with the component Lagrangian density of the canonical dimension **6**. This amounts to the one-loop finiteness of $6D\,,\,\mathcal{N}=(1,1)$ SYM theory.
- The single-trace on-shell $\mathbf{d}=\mathbf{8}$ invariant is unique and admits a simple representation in $\mathcal{N}=(1,1)$ superspace

$$\mathcal{S}^{(1)}_{(8)}\sim\int d\hat{\zeta}^{(-4,\hat{-}4)}\,(\phi^{+\hat{+}})^4\,.$$

- The double-trace $\mathbf{d}=\mathbf{8}$ invariant is also unique and reads

$$\mathcal{S}^{(2)}_{(8)}\sim\int d\hat{\zeta}^{(-4,\hat{-}4)}\,\mathrm{Tr}(\phi^{+\hat{+}})^2\mathrm{Tr}(\phi^{+\hat{+}})^2\,.$$

Both invariants are $\mathcal{N}=(1,0)$-invariant only *on-shell*. Since the $6D,\mathcal{N}=(1,0)$ HSS supergraph techniques should yield expressions with *off-shell* $\mathcal{N}=(1,0)$ supersymmetry, the absence of such invariants indicates the two-loop finiteness of $6D\,,\,\mathcal{N}=(1,1)$ SYM theory. Such invariants could still appear in the finite part of the total effective action.

- In the $\mathbf{d} = \mathbf{10}$ case the dimensions of the admissible superfield Lagrangian densities are

$$[L] = 2\,, \quad [\mathcal{L}^{(+4,\hat{+}4)}] = 6\,, \quad [\mathcal{L}_I^{(0,\hat{+}4)}] = [\mathcal{L}_{II}^{(+4,0)}] = 4\,. \tag{-5}$$

- Two single-trace $\mathbf{d} = \mathbf{10}$ invariants can be constructed

$$S_{(10)} = \mathrm{Tr} \int d\hat{Z}\, \phi^{+\hat{+}} \phi^{-\hat{-}}\,,\; \mathcal{S}_{I(10)} = \mathrm{Tr} \int d\hat{\zeta}_I^{(0,\hat{-}4)}\, (\phi^{+\hat{+}})^2 (\phi^{-\hat{+}})^2\,.$$

- In fact, these two are equivalent up to a numerical coefficient. This can be shown by representing $d\hat{Z} = d\hat{\zeta}_I^{(0,\hat{-}4)} (D^{\hat{+}})^4$. The double-trace invariants can be defined only as integrals over $3/4$ analytic subspaces

$$\mathcal{S}^{(2)}_{(10)I} \sim \int d\hat{\zeta}_I^{(0,\hat{-}4)} \mathrm{Tr}\,(\phi^{+\hat{+}} \phi^{-\hat{+}}) \mathrm{Tr}\,(\phi^{+\hat{+}} \phi^{-\hat{+}})\,,$$

$$\mathcal{S}^{(2)}_{(10)II} \sim \int d\hat{\zeta}_{II}^{(-4,0)} \mathrm{Tr}\,(\phi^{+\hat{+}} \phi^{+\hat{-}}) \mathrm{Tr}\,(\phi^{+\hat{+}} \phi^{+\hat{-}})\,.$$

- The single-trace invariant is off-shell $\mathcal{N} = (1,0)$ invariant, so it can appear as a three-loop counterterm. The second double-trace invariant cannot appear because it is $\mathcal{N} = (1,0)$ on-shell supersymmetric. The first double-trace invariant is still off-shell $\mathcal{N} = (1,0)$ invariant, so its appearance is not forbidden and something more is needed to explain the absence of logarithmic divergences at three loops.

d=12 invariants

- This part of the talk is based on a recent paper S. Buyukli & E. Ivanov, 2105.05899 [hep-th].
- The dimensions of the relevant Lagrangians are as follows

$$[L] = 4\,, \quad [\mathcal{L}^{(+4,\hat{+}4)}] = 8\,, \quad [\mathcal{L}_{I}^{(0,\hat{+}4)}] = [\mathcal{L}_{II}^{(+4,0)}] = 6\,.$$

- It is impossible to construct, out of the elementary "bricks" $\phi^{\pm\hat{\pm}}, \phi^{\mp\hat{\pm}}$ and those obtained from them through the action of the covariant differential operators $\nabla^{\pm}_{a}, D^{\hat{+}a}, \nabla^{\hat{-}a}$ and ∇_{ab}, the gauge invariant and manifestly analytic superfield objects possessing the charges $(+4, \hat{+}4)$, $(+4, 0)$ or $(0, \hat{+}4)$ with the above dimensions.
- So we are left with the chargeless general bi-HSS superfield densities L of the dimension 4 as the only candidates for the $\mathbf{d} = \mathbf{12}$ invariants.
- One can construct invariants containing four superfield strengths and no derivatives at all, or containing lesser number of strengths (three and two) and the proper number of derivatives distributed between them.

- We start with the first kind of invariants. A'priory one can introduce ten chargeless Lagrangian densities composed from $\phi^{+\hat{+}}$ and various harmonic projections thereof:

$$J_1 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{+\hat{+}}\phi^{-\hat{-}}\phi^{-\hat{-}}\,,\;\; J_2 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{-\hat{-}}\phi^{+\hat{+}}\phi^{-\hat{-}}\,,$$
$$J_3 = \mathrm{Tr}\,\phi^{+\hat{-}}\phi^{+\hat{-}}\phi^{-\hat{+}}\phi^{-\hat{+}}\,,\;\; J_4 = \mathrm{Tr}\,\phi^{+\hat{-}}\phi^{-\hat{+}}\phi^{+\hat{-}}\phi^{-\hat{+}}\,,$$
$$I_1 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{+\hat{-}}\phi^{-\hat{+}}\phi^{-\hat{-}}\,,\quad I_2 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{-\hat{+}}\phi^{-\hat{-}}\phi^{+\hat{-}}\,,$$
$$I_3 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{-\hat{-}}\phi^{+\hat{-}}\phi^{-\hat{+}}\,,\quad I_4 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{-\hat{-}}\phi^{-\hat{+}}\phi^{+\hat{-}}\,,$$
$$I_5 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{+\hat{-}}\phi^{-\hat{-}}\phi^{-\hat{+}}\,,\quad I_6 = \mathrm{Tr}\,\phi^{+\hat{+}}\phi^{-\hat{+}}\phi^{+\hat{-}}\phi^{-\hat{-}}\,.$$

- Using the opportunity to integrate by parts with respect to harmonic derivatives and various harmonic constraints and relations like $\phi^{-\hat{-}} = \nabla^{--}\phi^{+\hat{-}} = \nabla^{\hat{-}\hat{-}}\phi^{-\hat{+}}, \nabla^{--}\phi^{-\hat{\pm}} = \nabla^{\hat{-}-}\phi^{\pm\hat{-}} = 0$, etc, we find a lot of relations between these densities (up to total derivatives):

$$I_1 = I_3 = I_4 = I_6 = -\frac{1}{2}J_2\,,\quad I_2 = I_5 = \frac{1}{2}J_2 - J_1\,,\quad J_3 = J_1\,,\; J_4 = J_2\,.$$

- We choose as independent J_1, J_2 and so construct two independent single-trace invariants without derivatives

$$S^{(1)}_{(12)I} \sim \mathrm{Tr}\int d\hat{Z}\,\phi^{+\hat{+}}\phi^{+\hat{+}}\phi^{-\hat{-}}\phi^{-\hat{-}}\,,\;\; S^{(1)}_{(12)II} \sim \mathrm{Tr}\int d\hat{Z}\,\phi^{+\hat{+}}\phi^{-\hat{-}}\phi^{+\hat{+}}\phi^{-\hat{-}}\,.$$

- Analogously, one can define two independent double-trace invariants:

$$S^{(2)}_{(12)I} \sim \int d\hat{Z}\, \mathrm{Tr}\,(\phi^{++\hat{+}}\phi^{++\hat{+}})\, \mathrm{Tr}\,(\phi^{--\hat{-}}\phi^{--\hat{-}})\,,$$

$$S^{(2)}_{(12)II} \sim \int d\hat{Z}\, \mathrm{Tr}(\phi^{++\hat{+}}\phi^{--\hat{-}})\, \mathrm{Tr}\,(\phi^{++\hat{+}}\phi^{--\hat{-}})\,.$$

- As for invariants with derivatives, one can handle them in a similar way. Only two independent invariants are left:

$$S^{(1)}_{(12)(3)} \sim \mathrm{Tr} \int d\hat{Z}\, D^{\hat{+}a}\phi^{--\hat{-}}\nabla^{+}_{a}\phi^{--\hat{-}}\phi^{++\hat{+}},$$

$$S^{(1)}_{(12)(2)} \sim \mathrm{Tr} \int d\hat{Z}\, \nabla_{ab}\phi^{--\hat{-}}\nabla^{ab}\phi^{++\hat{+}}\,.$$

We observe that the $\mathbf{d} = \mathbf{12}$ invariants cannot be transformed to integrals of the manifestly analytic Lagrangian densities over the appropriate analytic subspaces and $\mathcal{N} = (1,0)$ supersymmetry is off-shell in them. So exists no any "protection" for them to appear as possible 4-loop counterterms in the $\mathcal{N} = (1,0)$ superfield perturbation theory computations in $\mathcal{N} = (1,1)$ SYM.

Passing to $\mathcal{N} = (1, 0)$ SYM superfields

- Neglecting the hypermultiplet superfields yields

$$\phi^{+\hat{+}} \;\to\; -\theta^{\hat{+}}_a W^{+a}\,, \quad V^{\hat{+}\hat{+}} \;\to\; i\theta^{\hat{+}}_a \theta^{\hat{+}}_b \,\mathcal{A}^{ab}\,.$$
$$\phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}} \;\to\; -\theta^{\hat{+}}_a W^{-a}\,.$$

- It is also straightforward to compute the $\mathcal{N} = (1, 0)$ SYM limit of $V^{\hat{-}\hat{-}}$

$$V^{\hat{-}\hat{-}} \;\to\; i\theta^{\hat{-}}_a \theta^{\hat{-}}_b \,\mathcal{A}^{ab} - \frac{1}{6}\Psi^{\hat{-}3d}\theta^{\hat{+}}_a \mathcal{D}_d W^{+a} - \frac{1}{24}\Psi^{\hat{-}4}\,\theta^{\hat{+}}_a\theta^{\hat{+}}_b\,\{W^{+a}, W^{-b}\}$$

and, then, of $\phi^{\pm\hat{-}} = \nabla^{\hat{-}\hat{-}}\phi^{\pm\hat{+}}$:

$$\begin{aligned}
\phi^{+\hat{-}} \;&\to\; -\theta^{\hat{-}}_a W^{+a} - i\theta^{\hat{-}}_b\theta^{\hat{-}}_c\theta^{\hat{+}}_a \mathcal{D}^{bc} W^{+a} - \frac{1}{6}\Psi^{\hat{-}3d}\theta^{\hat{+}}_a\theta^{\hat{+}}_b D^{+}_d\{W^{-a}, W^{+b}\} \\
&\quad + \frac{1}{24\cdot 6}\Psi^{\hat{-}4}\Psi^{\hat{+}3d}\varepsilon_{abcd}[\{W^{+a}, W^{-b}\}, W^{+c}]\,, \\
\phi^{-\hat{-}} \;&\to\; -\theta^{\hat{-}}_a W^{-a} - i\theta^{\hat{-}}_b\theta^{\hat{-}}_c\theta^{\hat{+}}_a \mathcal{D}^{bc} W^{-a} - \frac{1}{6}\Psi^{\hat{-}3d}\theta^{\hat{+}}_a\theta^{\hat{+}}_b [D^{+}_d W^{-a}, W^{-b}] \\
&\quad + \frac{1}{24\cdot 6}\Psi^{\hat{-}4}\Psi^{\hat{+}3d}\varepsilon_{abcd}[\{W^{+a}, W^{-b}\}, W^{-c}]\,.
\end{aligned}$$

- To pass to $\mathcal{N}=(1,0)$ superfields, one represent the $\mathcal{N}=(1,1)$ integration measure as $d\hat{Z}=dZd\hat{u}(D^{\hat{-}})^4(D^{\hat{+}})^4$ and then integrate over $\theta^{\hat{\pm}a}$ and $u_A^{\hat{\pm}}$. In this way, we obtain

$$S^{(1)}_{(12)I,II} \to \int dZ\, L^{(1)}_{(12)I,II}\,,$$
$$L^{(1)}_{(12)I} = -2\varepsilon_{abcd}\mathrm{Tr}\{\big(W^{-a}W^{+b}W^{+c}W^{-f} + W^{-f}W^{+a}W^{+b}W^{-c}\big)D^{+}_{f}W^{-d}$$
$$+ W^{+a}W^{+b}\mathcal{D}_{fg}W^{-c}\mathcal{D}^{fg}W^{-d}\},$$
$$L^{(1)}_{(12)II} = 2\varepsilon_{abcd}\mathrm{Tr}\{\big(W^{-a}W^{+b}W^{-f}W^{+c} + W^{+a}W^{-f}W^{+b}W^{-c}\big)D^{+}_{f}W^{-d}$$
$$- W^{+a}\mathcal{D}_{fg}W^{-b}W^{+c}\mathcal{D}^{fg}W^{-d}\}.$$

- For double-trace invariants we obtain

$$S^{(2)}_{(12)I,II} \to \int dZ\, L^{(2)}_{(12)I,II}\,,$$
$$L^{(2)}_{(12)I} = 4\,\varepsilon_{abcd}\mathrm{Tr}\big(W^{+a}\mathcal{D}_{gf}W^{+b}\big)\mathrm{Tr}\big(W^{-c}\mathcal{D}^{gf}W^{-d}\big),$$
$$L^{(2)}_{(12)II} = -2\,\varepsilon_{abcd}\{\mathrm{Tr}\big(W^{+a}\mathcal{D}_{fg}W^{-b}\big)\mathrm{Tr}\big(W^{+c}\mathcal{D}^{fg}W^{-d}\big)$$
$$+\mathrm{Tr}\big(W^{-f}W^{+a}\big)\mathrm{Tr}\big(\{W^{+b},W^{-c}\}D^{+}_{f}W^{-d}\big)\}.$$

- In the abelian limit all these invariants coincide up to numerical coefficients,

$$S^{(\mathrm{abel})}_{(12)} \sim \varepsilon_{abcd} \int dZ \, \partial^{gf} W^{+a} \partial_{gf} W^{+b} W^{-c} W^{-d} \,,$$

which is not a total derivative even on shell, with $\partial_{gf}\partial^{gf} W^{\pm d} = 0$. This sort of invariants could be relevant to the Coulomb branch of the theory, with the original gauge symmetry broken down to some abelian subgroup.

- For invariants with derivatives, the reduction to their $\mathcal{N} = (1, 0)$ SYM cores can also be performed. The characteristic feature of the resulting expressions is that they are vanishing in the abelian limit.

- It is of interest to see what kind of interaction the abelian invariant gives in components, with all fields besides gauge field being omitted. This further reduction amounts to representing

$$V^{++} = i\theta^{+a}\theta^{+b}A_{ab}\,,$$

$$V^{--} = i\theta^{-a}\theta^{-b}A_{ab} + \Psi^{-3}_{d}\theta^{+c}\mathcal{F}^{d}_{c}\,,\;\mathcal{F}^{d}_{c} := \frac{1}{6}\varepsilon^{abfd}\left(\partial_{ab}A_{fc} - \partial_{fc}A_{ab}\right),$$

$$W^{+a} = -\theta^{+b}\mathcal{F}^{a}_{b}\,,\;W^{-a} \to -i\theta^{-b}\theta^{-c}\theta^{+d}\partial_{bc}\mathcal{F}^{a}_{d}\,.$$

- Substituting it in $S^{(\text{abel})}_{(12)}$ and integrating there over θ s we obtain

$$S^{(\text{abel})}_{(12)} \to -2\varepsilon^{abcd}\varepsilon_{efgh}\int d^6x\,\left(\partial^{lt}\mathcal{F}^{e}_{a}\partial_{lt}\mathcal{F}^{f}_{b}\right)\left(\partial^{mn}\mathcal{F}^{g}_{c}\partial_{mn}\mathcal{F}^{h}_{d}\right).$$

  In the vector notation, with

$$\mathcal{F}^{a}_{b} = -\frac{1}{12}\,(\sigma^{MN})^{a}_{b}\mathcal{F}_{MN}\,,\quad \mathcal{F}_{MN} = \partial_M A_N - \partial_N A_M\,,$$

  this expression can be rewritten as

$$S^{(\text{abel})}_{(12)} \sim \int d^6x\Big[\left(\partial\mathcal{F}^{MN}\cdot\partial\mathcal{F}_{MN}\right)^2 + 2\left(\partial\mathcal{F}^{MN}\cdot\partial\mathcal{F}^{ST}\right)\left(\partial\mathcal{F}_{MN}\cdot\partial\mathcal{F}_{ST}\right)$$
$$-4\left(\partial\mathcal{F}^{MN}\cdot\partial\mathcal{F}^{ST}\right)\left(\partial\mathcal{F}_{MS}\cdot\partial\mathcal{F}_{NT}\right) - 8\left(\partial\mathcal{F}^{M}{}_{N}\cdot\partial\mathcal{F}_{MT}\right)\left(\partial\mathcal{F}^{SN}\cdot\partial\mathcal{F}_{S}{}^{T}\right)\Big],$$

  with $\partial\mathcal{F}^{MN}\cdot\partial\mathcal{F}_{MN} := \partial^L\mathcal{F}^{MN}\partial_L\mathcal{F}_{MN}$, etc.

It is interesting that there exist no $\mathbf{d = 12}$ invariants which could produce, in the abelian limit, the expression like $\sim F^6$, as distinct from the $\mathbf{d = 8}$ invariants, which contain F^4 term in such a limit.

Summary and outlook

- Off-shell $\mathcal{N} = (1, 0)$ and on-shell harmonic $\mathcal{N} = (1, 1)$ superspaces can be efficiently used to construct higher-dimensional invariants in $6D$ SYM theories.

- The full set of superfield invariants of dimensions $\mathbf{d} = \mathbf{8}$, $\mathbf{d} = \mathbf{10}$ and $\mathbf{d} = \mathbf{12}$ with $\mathcal{N} = (1, 1)$ on-shell supersymmetry was explicitly given.

- All $\mathbf{d} = \mathbf{6}$ $\mathcal{N} = (1, 1)$ invariants are on-shell vanishing, implying the UV finiteness of $\mathcal{N} = (1, 1)$ SYM at one loop.

- No $\mathbf{d} = \mathbf{8}$ $\mathcal{N} = (1, 1)$ invariants with off-shell $\mathcal{N} = (1, 0)$ supersymmetry exist whence the two-loop finiteness follows.

- The single-trace $\mathbf{d} = \mathbf{10}$ invariant can be given as an integral over full $\mathcal{N} = (1, 1)$ superspace, while the double-trace one cannot. This feature combined with an additional reasoning could explain why the double-trace invariant is UV protected.

- Four independent single-trace and two double-trace $\mathcal{N} = (1, 1)$ invariants with canonical dimension $\mathbf{d} = \mathbf{12}$ exist, all admitting off-shell $\mathcal{N} = (1, 0)$ supersymmetry. They could appear as 4-loop counterterms.

- *Some further possible lines of development*:

(a) To reproduce all the higher-dimensional invariants constructed from the quantum $\mathcal{N} = (1, 0)$ superfield perturbation theory.

(b) To move farther to the dimensions $\mathbf{d} \geq \mathbf{14}$. There are arguments (Smilga, 2016) that, starting with $\mathbf{d} = \mathbf{16}$ (six loops), the equations of motion are deformed and so the on-shell $\mathcal{N} = (1, 1)$ approach should be properly modified. Which principle could be behind such a modification?

(c) To work out the quantum superfield perturbation theory directly in $\mathcal{N} = (1, 1)$ double-harmonized superspace.

(d) To apply the same methods for constructing the Born-Infeld action with manifest off-shell $\mathcal{N} = (1, 0)$ and hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetries. To check the hypothesis that such an action could be identified with the full quantum effective action of $\mathcal{N} = (1, 1)$ SYM.

(e) Applications in supergravity?

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THANK YOU!