Quantum codes and conformal field theories

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Plan of the talk

- Introduction to codes: classical and quantum codes, connection to lattices and CFTs

- Applications: code theories and modular bootstrap

- Applications: code theories and holography

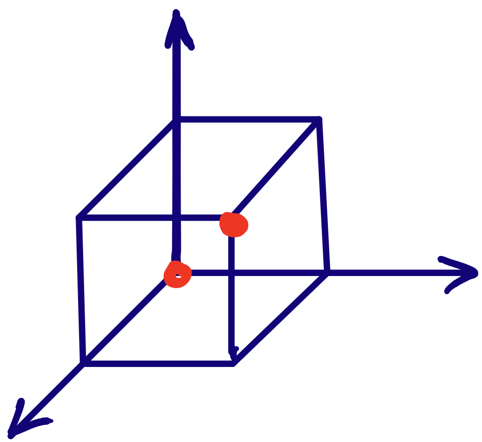
What is a code?

- linear binary $[n, k, d]$ code – 2^k codewords, each is a binary string of length n

  - k-dimensional vector subspace in n dimensional space, over field $\mathbb{Z}_2$

  - 2^k "highlighted" vertexes of a unit n-dimensional cube

  - Hamming distance d – minimal distance between codewords measured with ant norm



$c_0 = (0, 0, 0)$

$c_1 = (1, 1, 1)$

$[3, 1, 3]$

Basics of code theory

- dual code $\mathcal{C}^*$, dual to $\mathcal{C}$ in the sense of linear algebra
- spectrum of code, enumerator polynomial

$$W_{\mathcal{C}}(x,y) = \sum_{c\in\mathcal{C}} x^{n-|c|} y^{|c|}$$

for self-dual code $W_{\mathcal{C}}(x,y) = W_{\mathcal{C}}(x+y, x-y)/2^{n/2}$

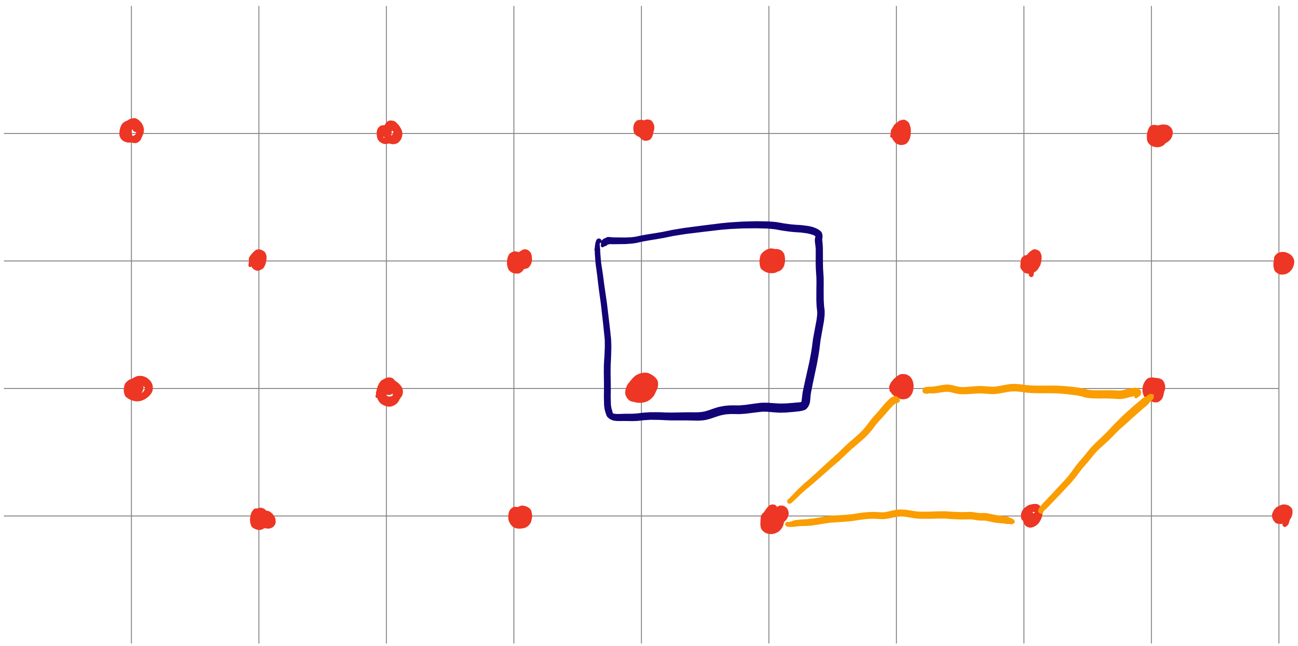
for double-even code, $|c| \vdots 4$, $W_{\mathcal{C}}(x,y) = W_{\mathcal{C}}(x, iy)$

Codes and lattices

$c_0 = (0,0)$

$c_1 = (1,1)$

- Construction A: code defines lattice, $x/\sqrt{2}$, $x \in \mathbb{Z}^n$ is a lattice vector iff $x \bmod 2 \in \mathcal{C}$



lattice theta-function $\Theta_\Lambda = \sum_{x\in\Lambda} e^{i\tau|x|^2} = W(\theta_3(2\tau), \theta_2(2\tau))$

Codes, lattices, and CFTs

- double-even self-dual code $\mathcal{C}$ defines even self-dual lattice
- even self-dual lattice defines chiral CFT

$$Z_{\rm CFT}(\tau) = \frac{W_{\mathcal{C}}(\theta_3(2\tau), \theta_2(2\tau))}{\eta(\tau)^n}$$

modular invariance of $Z_{\rm CFT}(\tau)$ reduces to algebraic properties of $W_{\mathcal{C}}$

$$W(x,y) = W(x,iy), \quad W(x,y) = W(x+y, x-y)/2^{n/2}$$

any homogeneous positive $W(x,y)$ satisfying these identities defines modular-invariant Z

Quantum codes

- stabilizer codes

$$g_i \Psi = \Psi, \qquad g_i = g(c_i) \equiv \sigma_{c_i^1} \otimes \cdots \otimes \sigma_{c_i^n}$$

vectors $c \in \mathbb{F}_4^n$ span linear subspace $\mathcal{C} \subset \mathbb{F}_4^n$

- quantum codes as classical codes

$\mathbb{F}_4$ includes four elements: $0, 1, \omega = e^{2\pi i/3}, \bar{\omega} = e^{-2\pi i/3}$

and the identity $2\,x = 0$ for any $x \in F_4$

$$g(c_1)g(c_2) \propto g(c_1 + c_2)$$

Real self-dual quantum codes

- enumerator polynomial

$$W(t,x,y,z) = \sum_{c\in\mathcal{C}} t^{n-\mathrm{w}(c)} x^{\mathrm{w}_x(c)} y^{\mathrm{w}_y(c)} z^{\mathrm{w}_z(c)}$$

- self-dual code: $\dim \mathcal{C} = n$

$$W(t,x,y,z) = W(t+x+y+z, t+x-y-z, t-x+y-z, t-x-y+z)/2^n$$

- real code: $\mathrm{w}_y(c) \vdots 2$

$$W(t,x,y,z) = W(t,x,-y,z)$$

Narain theories

- compactification of n scalars on a torus

$$S = \int \partial\phi^I \bar{\partial}\phi^J G_{IJ} + \partial\phi^I \wedge \bar{\partial}\phi^J B_{IJ}$$

Narain theories are pamaterized by Narain lattices Λ – even self-dual lattices in $\mathbb{R}^{n,n}$

$$\Lambda^T g \Lambda = g, \qquad \Lambda \in O(n,n)$$

- Narain CFT parition function

$$Z = \frac{\Theta_\Lambda}{|\eta|^{2n}}, \qquad \Theta_\Lambda = \sum_{(p_L,p_R)\in\Lambda} q^{p_L^2/2}\,\bar{q}^{p_R^2/2}$$

Quantum codes, Narain lattices, and CFTs

- self-dual real stabilizer code $\mathcal{C}$ defines Narain lattice – even self-dual lattice in $\mathbb{R}^{n,n}$

  Gray map $F_4 \leftrightarrow \mathbb{Z}_2^2$

  $$0 \leftrightarrow (0,0), \quad \omega \leftrightarrow (1,0), \quad \bar{\omega} \leftrightarrow (0,1), \quad 1 \leftrightarrow (1,1)$$

- Narain lattice defines Narain CFT

  $$Z_{\rm CFT}(\tau,\bar{\tau}) = \frac{W_{\mathcal{C}}(a,b,c,d)}{|\eta(\tau)|^{2n}}$$

  modular invariance of $Z_{\rm CFT}(\tau)$ reduces to algebraic properties of $W_{\mathcal{C}}$

- code automorphism groug (Clifford group) $\supset$ T-duality

  $\Psi \to u_1 \otimes \cdots \otimes u_n \Psi$

  stabilizer code $\Rightarrow$ graph code $g_i = \sigma_z^i \prod_{j=1}^n (\sigma_x)^{B_{ij}}$

  ELC graph equivalence = T-duality

Quantum codes and modular bootstrap

- quantum codes define a special sub-family of Narain theories – code CFTs
- code CFTs can be parametrized by graphs

  many examples of isospectral theories, starting from $n = 7$
  analog of Milnor's example
- relation to codes provides an "anzats" to solve modular bootstrap constraints

  optimal theories for $n = c = 3, 4, 5,$

  among them non-chiral E8 with $c = n = 4$ and $\Delta_1 = 1$

  many examples of "fake" W and $Z(\tau, \bar{\tau})$

with A. Shapere, PRL 126, 161602 (2021)
with A. Shapere, JHEP 160 (2021)

Code theories and holography

- Ensemble of Narain theories dual to "U(1)-gravity" (CS theory) in the bulk
  Afkhami-Jeddi, Cohn, Hartman, Tajdini'20, Maloney-Witten'20, Datta, Duary, Kraus, Maity, Maloney'21, Benjamin,Keller, Ooguri, Zadeh'21, Meruliya, Mukhi, Singh'21, Ashwinkumar et al.'21

- Code theories – new playground to mimic properties of a holographic ensemble

  microscopic bulk description for code theories?

Support for classical description in the bulk

- new bound on binary Hamming distance

$$0.31\,(\text{binary self}-\text{dual}) < d_b/n \leq 0.36 < 0.42\,(\text{binary isodual})$$

- variance of density of states is exponentially suppressed by central charge

$$(\overline{\rho^2}-\bar{\rho}^2) = e^{-O(c)}\bar{\rho}^2$$

- for code CFTs

- for Euclidean even self-dual lattices / chiral CFTs

with A. Shapere'2012.15830

Conclusions

- connection between quantum codes and CFTs provides a new tool to study individual theories as well as ensembles

  - optimal theories (with maximal spectral gap), isospectral theories, "fake" Z, ansatz to solve modular bootstrap

  - exponentially small variance $\overline{\delta\rho^2} \sim e^{-O(c)}\,\overline{\rho}^2$, distribution of d

- physical meaning of quantum codes, relation to holography?

- extension beyond Narain theories?