ON A GAUGE-INVARIANT DEFORMATION OF A CLASSICAL GAUGE-INVARIANT THEORY

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Goal

Description of the most general gauge invariant deformation of classical gauge invariant action.

Based on
I.L.B and P.M. Lavrov, arXiv:2104.11930 [hep-th].

Modern development of classical and quantum field theory is largely associated with the construction of the new gauge field theories. I want to point out that from formal point of view, the problem of constructing the new gauge theories can be treated as some kind of deformation problem for known gauge theories. I will show that such deformations can be described in general form within the Batalin-Vilkovisky (BV)-formalism on the base of master equation.

The possible applications:

- Construction of the interacting higher spin field models
- Construction of new extended supersymmetric field theories
- Construction of new models of quantum gravity

Motivations

Master equation has been introduced and studied by I.A. Batalin, G.A. Vilkovisky (1981, 1983, 1984). Aspects of gauge invariant deformation of given classical gauge invariant action were studied within BV formalism some time ago (G. Barnich, N. Boulanger, M. Henneaux, ...) in form of perturbation expansion in coupling constant. It was shown that solution to this equation is reduced to cohomological problem for some kind of BRST operator. I want to suggest another approach that does not require the expansions and does not use the cohomological analysis. It is proved that the general deformation of classical action is described in explicit form in terms of two arbitrary generating functions.

Plan

- Brief review of BV formalism
- Construction of deformed action
- Deformation of gauge symmetry
- Test: Yang-Mills theory as deformation of the Abelian gauge theory
- Application: local cubic interaction vertex for massless integer higher spin fields as the deformation of the free theory
- Summary

Brief review of BV formalism

Basic references: I.A. Batalin, G.A. Vilkovisky, Phys. Lett, 1981; Phys. Rev. D, 1983; Nucl. Phys. B, 1984.

The BV formalism was developed for covariant quantization of general gauge theories. The formalism operates, besides the basic fields, with the ghost fields and some number of auxiliary fields. It allows to introduce and explore some additional symmetry.

In my talk, I discuss the application of the BV formalism to the description of the deforming the classical gauge-invariant action with preservation of gauge invariance. To be more precise, let we have some given gauge invariant action. How to construct a family of the gauge invariant actions, which are obtained by gauge invariant deformation from given one and depend on the same set of fields.

General gauge theory is given by (in terms of Brice DeWitt condensed notations)

- Set of fields A^i;
  Grassmann parity $\varepsilon(A^i) = \varepsilon_i$, ghost number $\mathrm{gh}(A^i) = 0$
- Action $S_0[A]$
- Gauge generators $R^i_\alpha(A)$;
  $\varepsilon(R^i_\alpha(A)) = \varepsilon_i + \varepsilon_\alpha,\ \mathrm{gh}(R^i_\alpha(A)) = 0$
- Gauge transformations $\delta A^i = R^i_\alpha(A)\xi^\alpha$;
  gauge parameters ξ^α, $\varepsilon(\xi^\alpha) = \varepsilon_\alpha$
- Gauge invariance, $S_0[A]\overleftarrow{\partial}_{A^i} R^i_\alpha(A) = 0$

Assumption about gauge transformations

- The fields A^i are linear independent with respect to the index i.
- Gauge transformations are irreducible and form the closed gauge algebra

$$R^i_{\alpha,j}(A)R^j_\beta(A) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}(A)R^j_\alpha(A) = -R^i_\gamma(A)F^\gamma_{\alpha\beta}(A),$$

$$R^i_{\alpha,j}(A) = R^i_\alpha(A)\overleftarrow{\partial}_{A^j}$$

Fields ϕ^A and antifields ϕ^*_A

- $\phi^A = (A^i, C^\alpha), \quad \phi^*_A = (A^*_i, C^*_\alpha)$
- C^α $\big(\varepsilon(C^\alpha) = \varepsilon_\alpha + 1,\ \mathrm{gh}(C^\alpha) = 1\big)$
- Properties of the ghost and antighost fields

$$\varepsilon(\phi^*_A) = \varepsilon(\phi^A) + 1, \quad \mathrm{gh}(\phi^*_A) = -1 - \mathrm{gh}(\phi^A)$$

Master equation

- Basic object is extended action $S = S[\phi, \phi^*]$
- Master equation

$$(S, S) = 0$$

- Antibracket for any functionals $F[\phi, \phi^*]$ and $H[\phi, \phi^*]$

$$(G, H) = G\left(\overleftarrow{\partial}_{\phi^A}\overrightarrow{\partial}_{\phi^*_A} - \overleftarrow{\partial}_{\phi^*_A}\overrightarrow{\partial}_{\phi^A}\right)H$$

- Boundary condition

$$S[\phi, \phi^*]\Big|_{\phi^*=0} = S_0[A]$$

- Gauge invariance of initial action leads to invariance of extended action under the global BRST transformations

$$\delta_B S = 0,$$

where

$$\delta_B \phi^A = (\phi^A, S)\mu = \overrightarrow{\partial}_{\phi^*_A} S\,\mu, \quad \delta_B \phi^*_A = 0$$

Brief review of BV formalism

- Partial solution to master equation $S[\phi, \phi^*]$ up to linear in antifields terms has the form

$$\bar{S} = S_0[A] + A_i^* R^i_\alpha(A) C^\alpha - \frac{1}{2} C^*_\gamma F^\gamma_{\alpha\beta}(A) C^\beta C^\alpha (-1)^{\varepsilon_\alpha} + O(\phi^{*\,2}).$$

  Remark: in Yang-Mills theory S is the gauge invariant action $S_0[A]$ plus the BRST invariant functionals with the corresponding sources. In this case the master equation is known Zinn-Justin equation. The sources to BRST invariant functional play a role of antifields.

- Our aim is to describe a general form of arbitrary solution to the master equation with given boundary conditions in the sector of the fields ϕ^A, i.g. to find $\tilde{S}[\phi] = S[\phi, \phi^*]|_{\phi^*=0.}$ and describe the arbitrariness in such a solution.
- Action $\tilde{S}[\phi]$ is obtained by deformation of initial action S_0. We will call it the deformed action.
- Method: invariance of the antibracket under the anticanonical transformations.
  Remark: the antibracket is analogous to Poisson bracket in classical mechanics and the anticanonical transformations are analogous to canonical transformations preserving the Poisson bracket.

Construction of deformed action

- General solution of the master equation is described in terms of anticanonical transformation of fields and antifields $\phi \to \Phi(\phi, \phi^*)$, $\phi^* \to \Phi^*(\phi, \phi^*)$
- Anticanonical transformation is described by the generating functional $Y[\phi, \Phi^*]$ $(\varepsilon(Y) = 1,\ \mathrm{gh}(Y) = -1)$
- New fields and antifields are the solutions to equations

$$\phi^*_A = Y[\phi, \Phi^*]\overleftarrow{\partial}_{\phi^A}, \quad \Phi^A = \overrightarrow{\partial}_{\Phi^*_A} Y[\phi, \Phi^*]$$

- The transformed action in terms of initial fields and antifields is written as follows

$$\widetilde{S}[\phi, \phi^*] = S[\Phi(\phi, \phi^*), \Phi^*(\phi, \phi^*)]$$

with $S[\Phi(\phi, \phi^*), \Phi^*(\phi, \phi^*)] = S[\phi, \phi^*]\big|_{\phi \to \Phi(\phi,\phi^*), \phi^* \to \Phi^*(\phi,\phi^*)}$ and $S[\phi, \phi^*]$ is the given partial solution.
- The action $\widetilde{S}[\phi, \phi^*]$ satisfies to classical master equation $(\widetilde{S}, \widetilde{S}) = 0$
- Action is invariant under the BRST transformations
$\delta_B \widetilde{S} = 0, \quad \delta_B \phi^A = (\phi^A, \widetilde{S})\mu = \overrightarrow{\partial}_{\phi^*_A} \widetilde{S}\, \mu, \quad \delta_B \phi^*_A = 0$

General scheme of construction for Y.

- $Y[\phi, \Phi^*] = \Phi^*_A \phi^A + X[\phi, \Phi^*]$.
  First term means identical transformation, second one is responsible for deformation.
- X is given in form of expansion in antifields. Since the antifields will be switched off afterwards, it is sufficient to write

$$X[\phi, \Phi^*] = \Phi^*_A H^A(\phi) + \frac{1}{2}\Phi^*_A \Phi^*_B H^{BA}(\phi) + O(\phi^{*\,3})$$

  with some unknown arbitrary functions $H^A(\phi)$ and $H^{BA}(\phi)$.
- One the base of above relations and properties of the generating functional Y, one can prove

$$\Phi^A(\phi, \phi^*) = \phi^A + H^A(\phi) + \Phi^*_B(\phi, \phi^*) H^{BA}(\phi) + O(\phi^{*\,2})$$

  and

$$\Phi^*_A(\phi, \phi^*)\Big|_{\phi^*=0} = 0$$

Construction of deformed action

Deformed action $\widetilde{S}[\phi]$:

$$\widetilde{S}[\phi] = \widetilde{S}[\phi, \phi^*]\Big|_{\phi^*=0} = S[\Phi(\phi, \phi^* = 0), 0] = S[\phi + H(\phi), 0] = S_0[A + h(A)]$$

This is an maximally possible form for deformed action! Arbitrariness in deformed action is determined by the arbitrary function $h^i(A)$.

- Deformed action depends on the same fields as the initial action.
- Deformed action is gauge invariant by construction in terms of deformed generators.

The general structure of the deformed action is extremely simple. The arbitrary deformation of action is described by a simple shift of the field A^i in the initial action $S_0[A]$ by arbitrary functions $h^i(A)$.

Construction of deformed action

Discussion of the result for deformed action $\widetilde{S}[A] = S_0[A + h(A)]$.

- Is deformation the trivial field redefinition? Yes it is, if it is local.
- In general, the functions h^i can be non-local.
- The deformed action in general must be non-local functional.

There can be such a situation that general non-local deformed action admits a closed local sector. It means that the deformed action has the following structure $S_0[A + h(A)] = S_1[A] +$ *non-local terms*, where the action $S_1[A]$ is a local functional. If the deformed gauge transformations contain a local piece that leaves the action $S_1[A]$ invariant, we obtain the closed local sector of deformed theory.

Let for example the initial Lagrangian has a form $\mathcal{L}_0 \sim A\Box A$ and let $h \sim \frac{1}{\Box}F(A)$ with some local function $F(A)$. Then after some transformations the deformed Lagrangian takes the form $\tilde{\mathcal{L}} \sim A\Box A + 2F(A) +$ *non-local terms*. If deformed gauge transformations have a local piece leaving the Lagrangian $\tilde{\mathcal{L}}_1 \sim A\Box A + 2F(A)$ invariant, one gets a closed local sector of non-local theory.

Deformation of gauge symmetry

The deformation of the gauge generators is calculated on the based of the following considerations:

- Denoting the new fields as $\Phi^A = (\mathcal{A}^i, \mathcal{C}^\alpha)$ and using the relation $\Phi^A = \phi^A + H^A(\phi)$ derived before, one can write $\mathcal{A}^i = A^i + h^i(\phi),\ \mathcal{C}^\alpha = C^\alpha + g^\alpha(\phi)$ with two arbitrary functions $h^i(\phi)$ and $g^\alpha(\phi)$.
- Since $\mathrm{gh}(A) = 0$ and $\mathrm{gh}(C) = 1$ one can write $h^i(\phi) = h^i(A),\ g^\alpha(\phi) = g^\alpha{}_\beta(A)C^\beta$.
- Master equation $(S, S) = 0$ and the relation $S[A] = S_0[A + h]$ lead to

$$S[A]\overleftarrow{\partial}_{A^j}(M^{-1}(A))^j{}_i R^i_\alpha(\widetilde{A})M^\alpha{}_\beta(A) = 0,$$

  where

$$M^j{}_i(A) = \delta^j{}_i + h^j(A)\overleftarrow{\partial}_{A^i},\ M^\alpha{}_\beta(A) = \delta^\alpha{}_\beta + g^\alpha{}_\beta(A).$$

Denoting

$$\mathbf{R}^i_\alpha(A) = (M^{-1}(A))^j{}_i R^i_\alpha(\widetilde{A}) M^\alpha{}_\beta(A),$$

we rewrite

$$S[A]\overleftarrow{\partial}_{A^j}(M^{-1}(A))^j{}_i R^i_\alpha(\widetilde{A}) M^\alpha{}_\beta(A) = 0,$$

as

$$S[A]\overleftarrow{\partial}_{A^j}\mathbf{R}^j_\alpha(A) = 0.$$

This relation is condition of gauge invariance of the deformed action $S[A]$ in terms of deformed generators $\mathbf{R}^i_\alpha(A)$.

Deformation of gauge symmetry

Direct calculations lead to gauge algebra for deformed generators

$$\big(\mathbf{R}^i_\alpha(A)\overleftarrow{\partial}_{A^j}\big)\mathbf{R}^j_\beta(A) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}\big(\mathbf{R}^i_\beta(A)\overleftarrow{\partial}_{A^j}\big)\mathbf{R}^j_\alpha(A) = -\mathbf{R}^i_\gamma(A)\mathbf{F}^\gamma_{\alpha\beta}(A)$$

with deformed structure coefficients

$$\begin{aligned}\mathbf{F}^\gamma_{\alpha\beta}(A) \;=\; & -(M^{-1})^\gamma{}_\lambda(A)\widetilde{F}^\lambda_{\rho\sigma}(A)M^\sigma{}_\alpha(A)M^\rho{}_\beta(A)(-1)^{\varepsilon_\alpha\varepsilon_\rho} - \\ & -\big((M^{-1})^\gamma{}_\mu(A)\overleftarrow{\partial}_{A^j}\big)\mathbf{R}^j{}_\alpha(A)M^\mu{}_\beta(A)(-1)^{\varepsilon_\alpha\varepsilon_\mu} + \\ & +\big((M^{-1})^\gamma{}_\mu(A)\overleftarrow{\partial}_{A^j}\big)\mathbf{R}^j{}_\beta(A)M^\mu{}_\alpha(A)(-1)^{\varepsilon_\alpha(\varepsilon_\beta+\varepsilon_\mu)}.\end{aligned}$$

and $\widetilde{F}^\gamma_{\alpha\beta}(A) = F^\gamma_{\alpha\beta}(A+h)$.

- The deformed gauge algebra is closed as well as the initial gauge algebra.
- Even if the initial gauge theory is Abelian, the deformed gauge theory is non-Abelian in general.
- The arbitrary deformation in the initial gauge theory is defined by two (in general, non-local) arbitrary functions.

Consider the Abelian gauge theory

$$S_0[A] = -\frac{1}{4} F^a_{0\mu\nu}(A) F^{a\mu\nu}_0(A),$$

a is the index of some semi-simple Lie algebra with the structure constants f^{abc}. Action is invariant under the standard Abelian gauge transformations.

According to general result, the deformed action is $S[A] = S_0[A+h]$, where $h(A) = \{h^a_\mu(A)\}$ is an arbitrary function.

- Consider the following function

$$h^a_\mu(A) = \frac{1}{\Box}\Big[\frac{1}{2}\partial^\nu (f^{abc} A^b_\nu A^c_\mu) + \frac{1}{4} f^{abc} f^{cmn} A^{b\nu} A^m_\nu A^n_\mu\Big]$$

- $S_0[A+h] = S_{YM}[A]$ + *non-local terms containing the* $\frac{1}{\Box}$.
- $\mathbf{R}^{ab}_\mu(A) = D^{ab}_\mu(A) + R^{ab}_{1\mu}(A)$

Yang-Mills theory is a local sector of deformed non-local vector field theory.

Application: local cubic interaction vertex for massless integer higher spin fields as the deformation of the free theory

Action for free massless higher spin theory (Fronsdal, 1978)

$$\begin{aligned}S^{(2)}[\varphi] =&\int d^4x\{-\varphi_{\mu_1\ldots\mu_s}\Box\varphi^{\mu_1\ldots\mu_s} - \frac{s}{2}\partial_\alpha\varphi^{\alpha\mu_2\ldots\mu_s}\partial^\beta\varphi_{\beta\mu_1\ldots\mu_s}\\&-\frac{s(s-1)}{2}\varphi^\rho{}_{\rho\mu_3\ldots\mu_s}\partial_\alpha\partial_\beta\varphi^{\alpha\beta\mu_3\ldots\mu_s} -\\&-\frac{s(s-1)}{4}\partial_\alpha\varphi^\rho{}_{\rho\mu_3\ldots\mu_s}\partial^\alpha\varphi_\sigma{}^{\sigma\mu_3\ldots\mu_s}\\&-\frac{s(s-1)(s-2)}{8}\partial_\alpha\varphi_\rho{}^{\rho\alpha\mu_4\ldots\mu_s}\partial^\beta\varphi^\sigma{}_{\sigma\beta\mu_4\ldots\mu_s}\}.\end{aligned}$$

Here Here $\varphi_{\mu_1\ldots\mu_s}$ is a totally symmetric double traceless field. The theory is invariant under the the Abelian gauge transformations with the traceless totally symmetric parameters $\xi_{\mu_1\ldots\mu_{s-1}}$.

Application: local cubic interaction vertex for massless integer higher spin fields as the deformation of the free theory

Construction of deformed action with help of relation $S[A] = S_0[A + h(A)]$.

- Approach under consideration assumes that the fields and gauge parameters are unconstrained. However, in Fronsdal action the fields and parameters obey the traceless conditions and the approach can not be applied in the literal form. It requires a generalization of the approach. To simplify the situation, we will follow the considerations that are sometimes used in the theory of higher-spin fields when constructing the interaction vertices. Assume that there are no traceless conditions for the fields and parameters from the very beginning and then impose needed restrictions afterwards.
- Apply the relation $S[A] = S_0[A + h(A)]$ to Fronsdal action. To get the cubic vertex it is sufficient to take the function h to be quadratic in fields φ.
- Introduce the function h in the form

$$h^{\mu_1\dots\mu_s} = c_k g^{s+2k}\frac{1}{\Box}\partial^{\{\mu_1}\dots\partial^{\mu_k}\varphi^{\mu_{k+1}\dots\mu_s\}\nu_1\dots\nu_k}\partial_{\nu_1}\dots\partial_{\nu_k}\partial^{\lambda_1}\dots\partial^{\lambda_s}\varphi_{\lambda_1\dots\lambda_s}.$$

Here the parameter $k = 0, \dots s$; c_k are the arbitrary real constants and g is a coupling constant of the dimension $\dim(g) = -1$.

Application: local cubic interaction vertex for massless integer higher spin fields as the deformation of the free theory

Construction of deformed action with help of relation $S[A] = S_0[A + h(A)]$.

Substitution of the above h into Fronsdal action leads to the deformed action $S[\varphi] = S^{(2)}[\varphi] + S^{(3)}_{local}[\varphi] +$ *non-local terms.*

Here

$$S^{(3)}_{local}[\varphi] = -2c_k g^{s+2k} \int d^4x \varphi^{\mu_1 \dots \mu_s} \partial_{\{\mu_1} \dots \partial_{\mu_k} \varphi_{\mu_{k+1} \dots \mu_s\}\nu_1 \dots \nu_k} \times \\ \times \partial^{\nu_1} \dots \partial^{\nu_k} \partial_{\lambda_1} \dots \partial_{\lambda_s} \varphi^{\lambda_1 \dots \lambda_s}$$

This vertex describes the interaction of three fields with the same spins s and consistent with the known results for cubic vertex (see e.g. Metsaev, Phys. Lett B, 2013).

Last step is deriving the deformed gauge transformations. They can be found on the base of general deformed gauge generators and given function h. The deformed gauge transformations in general are non-local but they have a local part that leaves invariant the expression $S^{(2)} + S^{(3)}_{local}$ up to fourth order terms in φ. As a result, the deformed theory under consideration contains a closed local sector including cubic interaction vertex.

Summary

- We have described a general procedure of gauge invariant deformation of classical gauge invariant action. The procedure is based on use of the BV-formalism where the central object is the master equation. We have proved that the arbitrary gauge deformation of a given gauge invariant theory is described by two arbitrary generating functions.
- The deformation of initial action has extremely simple form and means a replacement in the initial action the gauge field A by the field $A + h(A)$ with arbitrary in general non-local generating function $h(A)$. The deformation of gauge generators is described by the same function $h(\phi)$ and another arbitrary function $g(A)$. Deformed algebra is closed as well as the initial algebra and contains the deformed structure coefficients. Note, that even if the initial theory is Abelian, the corresponding deformed theory is non-Abelian in general.

- The essential feature of the obtained deformed theory is that it should in general be non-local. However, in special cases there can be a situation when such a non-local theory contains a closed local sector. It means that the deformed action can involve the local piece which is invariant under the local piece of deformed gauge transformations.
- A first important test for the theory under consideration is a possibility to derive Yang-Mills theory. We have shown that if to start with Abelian gauge theory and apply the transformation $A \to A + h(A)$ with some special function $h^a_\mu(A)$ we obtain a non-local vector field theory with local piece in action which just is the Yang-Mills action. The corresponding piece in deformed gauge transformation is the Yang-Mills gauge transformation.
- We have shown that the developed deformation procedure allows to derive the local cubic interaction vertices for massless higher spin fields. In this case we begin with known Fronsdal free action and apply the transformation $\phi \to \phi + h(\phi)$ with appropriate non-local function $h(\phi)$.

Possible generalizations and developments

- Generalization for the gauge theories with open gauge algebra.
- Generalization for the gauge theories with dependent generators.
- Relations of the quantum effective actions for the classical gauge theories obtained one from another by deformation.
- Generation of complete general relativity by means of deformation of free massless spin 2 theory.
- Generation of interaction vertices for the higher spin fields.

THANK YOU VERY MUCH!