Non-abelian U-duality

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T-duality (abelian)

2d σ-model with a U(1) isometry along X^9 and added Lagrange term:

$$
\begin{aligned}
S &= \int_{\Sigma_2} *E^a \wedge E^b \eta_{ab} + B_{(2)} + \tilde{X} dA, \\
E^a &= \underbrace{E_\mu{}^a\, \partial_i X^\mu\, d\sigma^i}_{\mu=0,\dots,8} + E_9{}^a\, A_i\, d\sigma^i
\end{aligned} \tag{1}
$$

The standard T-duality algorithm:

- Integrate out $\tilde{X}$: $A_i = \partial_i X^9$, gives the initial σ-model
- Integrate out A_i: gives a T-dual σ-model
- Backgrounds are related by Buscher rules

[Buscher(87,88), Duff(90)]

T-duality (non-abelian)

Isometry group G acts transitively and freely.

$$[T_I, T_J] = f_{IJ}{}^K T_K. \tag{2}$$

σ-model with Lagrange multiplier is:

$$\begin{aligned} S &= \int_{\Sigma_2} *E^a \wedge E^b \eta_{ab} + B_{(2)} + \tilde{X}_I (dA^I + f_{JK}{}^I A^J \wedge A^K), \\ E^a &= \underbrace{E_\mu{}^a\, \partial_i X^\mu\, d\sigma^i}_{M} + \underbrace{E_I{}^a\, A_i{}^I\, d\sigma^i}_{G} \end{aligned} \tag{3}$$

The standard T-duality algorithm:

- Initial model: integrate out $\tilde{X}_I$: $A^I = (g^{-1}dg)^I$ for $g \in G$
- NAT dual model: integrate out $A^I{}_i$
- Backgrounds are related by Buscher rules

[Quevedo, de la Ossa (92)]

T-duality

- Abelian T-duality always generates backgrounds with U(1) isometry
- Map between fluxes, generation of non-geometric fluxes, exotic branes etc.
- NATD does not generate backgrounds with the same isometry G
- NATD serves as a solution generating technique: new backgrounds, nonCFT holography etc

Q: How to perform inverse non-abelian T-duality transformation?

A: Use classical Drinfeld double [Klimcik, Severa (95), von Unge (05)]

Poisson-Lie T-duality

$$S[X] = \int dz d\bar{z}\, E_{\mu\nu}\, \partial X^\mu \bar{\partial} X^\nu = \int dz d\bar{z}\, (G_{\mu\nu} + B_{\mu\nu})\, \partial X^\mu \bar{\partial} X^\nu \tag{4}$$

Right action of a group G on target (not necessarily isometric!):

$$\begin{gathered} S[X + \varepsilon^a v_a] - S[X] = \int dz d\bar{z}\, \varepsilon^a L_{v_a}(\mathcal{L}) + \int d\varepsilon^a \wedge J_a, \\ [v_a, v_b] = f_{ab}{}^c v_c, \\ J_a \ - \ \text{Noether currents} \end{gathered} \tag{5}$$

Possibilities:

- $dJ_a = 0$, $f_{ab}{}^c = 0$: abelian T-duality;
- $dJ_a = 0$, $f_{ab}{}^c \neq 0$: non-abelian T-duality;
- $dJ_a \neq 0$: non-commutative conservation law, Poisson-Lie T-duality,

Poisson-Lie T-duality

$v_a = v_a{}^\mu \partial_\mu$ generate isometries if

1. $dJ_a = \frac{1}{2}\tilde{f}_a{}^{bc}\, J_b \wedge J_c$ for some structure constants $\tilde{f}_a{}^{bc}$,
2. $L_{v_a}(\mathcal{L}) = \frac{1}{2}\tilde{f}_a{}^{bc}\, J_b \wedge J_c$

For $G_{\mu\nu}, B_{\mu\nu}$ this implies

$$L_{v_a}(E_{\nu\mu}) = \tilde{f}_a{}^{bc} v_b{}^\rho v_c{}^\sigma E_{\rho\mu} E_{\nu\sigma}. \tag{6}$$

Compatibility constraints:

$$4\tilde{f}_{[a}{}^{e[c} f_{b]e}{}^{d]} = f_{ab}{}^e \tilde{f}_e{}^{cd}. \tag{7}$$

Poisson-Lie T-duality

Dual model:

$$L_{\tilde{v}^a}(\tilde{E}_{\nu\mu}) = f_{bc}{}^a \tilde{v}^{b\rho} \tilde{v}^{c\sigma} \tilde{E}_{\rho\mu} \tilde{E}_{\nu\sigma}. \tag{8}$$

Double Drinfeld algebra

Algebraic structure behind Poisson-Lie T-duality: classical Drinfeld double $\mathcal{D} = (\mathfrak{g}, \tilde{\mathfrak{g}}, \eta)$ (Manin triple decomposition)

$$
\begin{aligned}
\text{bas } \mathfrak{g} &= \{T_a\}, \quad [T_a, T_b] = f_{ab}{}^c T_c, \\
\text{bas } \tilde{\mathfrak{g}} &= \{\tilde{T}^a\}, \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}_c{}^{ab} \tilde{T}^c, \\
\eta(T_a, \tilde{T}^b) &= \delta_a{}^b.
\end{aligned}
\tag{9}
$$

Compatibility constraint: $\mathfrak{g}$ (or $\tilde{\mathfrak{g}}$) is a bialgebra.

$$
[T_a, \tilde{T}^b] = \tilde{f}_a{}^{bc} T_c - f_{ac}{}^b \tilde{T}^c \tag{10}
$$

Different realisations of a given Drinfeld double by Manin triples correspond to Poisson-Lie dual backgrounds.

Double Drinfeld algebra

Natural $O(d,d)$ language for the Drinfeld double behind Poisson-Lie T-duality

- Combine $\{T_A\} = \{T_a, \tilde{T}^a\}$ to write the full algebra $\mathcal{D}$ as

$$[T_A, T_B] = \mathcal{F}_{AB}{}^C\, T_C, \quad \mathcal{F}_{ab}{}^c = f_{ab}{}^c, \quad \mathcal{F}_c{}^{ab} = \tilde{f}_c{}^{ab}. \tag{11}$$

- The quadratic form $\eta(T_A, T_B) = \eta_{AB}$

$$\eta_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

- The isotropic subgroup $\mathfrak{g}$ is picked by

$$\mathfrak{g} \otimes \mathfrak{g}\Big|_{1 \text{ of } O(d,d)} = 0. \tag{13}$$

Statements

- T-duality is a perturbative symmetry of string theory backgrounds (dimension 10);
- String on a torus $\mathbb{T}^d$ admits $O(d,d)$ T-duality symmetry group;
- Abelian and non-abelian T-dualities are special cases of more general Poisson-Lie T-duality;
- Poisson-Lie T-duality corresponds to different realisations of a given double Drinfeld algebra;
- There exists a U-duality extensions of the construction based on exceptional Drinfeld doubles.
- Abelian U-duality group for $11-d$-dimensional supergravity is $E_{d(d)}$.

M-theory

- At strong coupling $g_s \to \infty$ dynamics of string theory is described by the membrane interacting with 11-dimensional supergravity

$$G_{mn}, \quad C_{mnk} \tag{14}$$

- Supermembrane action is κ-symmetric if EoM's of 11d SUGRA are satisfied
[Bergshoeff, Sezgin, Townsend (1987)]

$$S_{11} = \int d^{11}x\sqrt{G}\Big(R[G] - \frac{1}{48}F_{mnkl}F^{mnkl} + \dots\Big) \tag{15}$$

- 11d sugra on $\mathbb{T}^d$ enjoys U-duality symmetry group $E_{d(d)}$
[Cremmer, Julia (1979), (1981)]

$$E_{5(5)} = SO(5,5), \quad E_{4(4)} = SL(5), \quad E_{3(3)} = SL(3)\times SL(2). \tag{16}$$

Exceptional Drinfeld algebra

- Generators fill some irrep $\mathcal{R}_1$ of $\mathrm{E}_{d(d)}$: $\{T_A\} = \{T_a, T^{a_1 a_2}, T^{a_1 \dots a_5}, \dots\}$

$$\begin{aligned} & d\,, && \mathcal{R}_1 = d \oplus \bar{d} && \text{of} \quad \mathrm{O(d,d)}\,, \\ & d = 4\,, && \mathcal{R}_1 = 10 && \text{of} \quad \mathrm{SL(5)}\,, \\ & d = 5\,, && \mathcal{R}_1 = \overline{16} && \text{of} \quad \mathrm{SO(5,5)}\,, \\ & d = 6\,, && \mathcal{R}_1 = 27 && \text{of} \quad \mathrm{E}_{6(6)}\,. \end{aligned} \tag{17}$$

- The algebra is defined by multiplication rule

$$T_A \circ T_B = \mathcal{F}_{AB}{}^C\, T_C \tag{18}$$

satisfying Leibniz identity

$$T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C + T_B \circ (T_A \circ T_C). \tag{19}$$

[Sakatani(19), Malek, Thompson(19), Malek, Thompson, Sakatani(07)]

Exceptional Drinfeld algebra

- The "isotropic" subalgebra $\mathfrak{g}$ is chosen by $\mathfrak{g}\otimes\mathfrak{g}\Big|_{\mathcal{R}_2}=0$

$$\begin{aligned} &d\,, && \mathcal{R}_2=1 && \text{of } O(d,d)\,,\\ &d=4\,, && \mathcal{R}_2=\overline{5} && \text{of } SL(5)\,,\\ &d=5\,, && \mathcal{R}_2=10 && \text{of } SO(5,5)\,,\\ &d=6\,, && \mathcal{R}_2=\overline{27} && \text{of } E_{6(6)}\,. \end{aligned} \tag{20}$$

- E.g. for O(d,d) $\mathfrak{g}=\mathrm{span}(T_a)$ can not compose the singlet. Equivalently $\tilde{\mathfrak{g}}=\mathrm{span}(\tilde{T}^a)$.
- $\mathfrak{g}$ is a Lie algebra: $T_a\circ T_b=f_{ab}{}^cT_c$ for $\{T_a\}=\mathrm{bas}\,\mathfrak{g}$
- $\mathfrak{g}$ acts on EDA by inherited adjoint action

[Sakatani(19), Malek, Thompson(19), Malek, Thompson, Sakatani(07)]

Simple example of $d = 4$ EDA

- Ten generators $\{T_a, T^{ab}\} \in \mathbf{10}$ of SL(5)
- Structure constants

$$\mathcal{F}_{AB}{}^{C} = \{f_{ab}{}^{c}, f_a{}^{bcd}, Z_a\} \tag{21}$$

- The algebra

$$\begin{aligned}
T_a \circ T_b &= f_{ab}{}^{c}\, T_c\,, \\
T_a \circ T^{b_1 b_2} &= f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,, \\
T^{a_1 a_2} \circ T_b &= -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} \\
&\quad - 9\, Z_c\, \delta^{[c}_{b}\, T^{a_1 a_2]}\,, \\
T^{a_1 a_2} \circ T^{b_1 b_2} &= -2\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c}\,,
\end{aligned} \tag{22}$$

Geometry

Non-linear realisation of a given EDA by exceptional geometry

1. Pick an isometric subalgebra $\mathfrak{g}$ and define $G = \exp \mathfrak{g}$ as

$$G \ni g = \exp h, \quad h = x^a T_a \in \mathfrak{g} \tag{23}$$

2. Compute adjoint action

$$\begin{aligned} g \cdot T_A \cdot g^{-1} &= T_A + h \circ T_A + \frac{1}{2} h \circ (h \circ T_A) + \dots \\ &= M_A{}^B T_B \end{aligned} \tag{24}$$

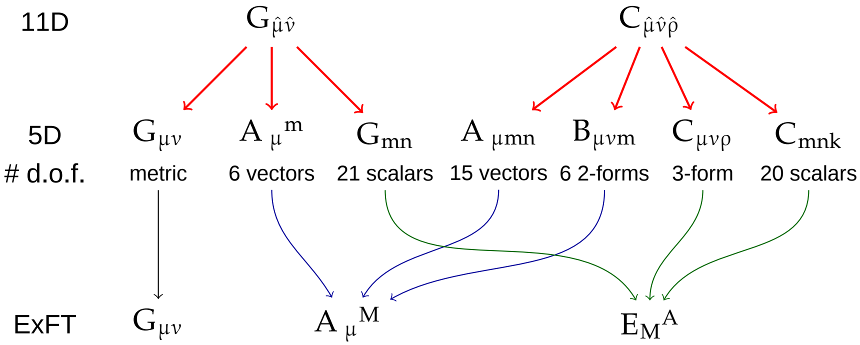
3. This defines a generalised frame field

$$E_A{}^M = M_A{}^B \mathbb{E}_B{}^M \in E_{d(d)}/K \tag{25}$$

E_6-covariant parametrisation

Field content

[Cremmer, Julia, Lue, Pope (97)]



E_6-covariant parametrisation

Generalised vielbein $E_M{}^A \in E_{6(6)} \times \mathbb{R}^+$

$$E_M{}^A = e^{\varphi} \begin{bmatrix} e_m{}^{\bar m} & -\frac{1}{\sqrt{2}} C_{m\,\bar n_1 \bar n_2} & \frac{1}{2} e_m{}^{\bar m} U + \frac{1}{4} C_{m k_1 k_2} V^{\bar n k_1 k_2} \\ 0 & e_{[\bar n_1}{}^{m_1} e_{\bar n_2]}{}^{m_2} & -\frac{1}{\sqrt{2}} V^{\bar m k_1 k_2} \\ 0 & 0 & e^{-1} e_n{}^{\bar n}, \end{bmatrix} \tag{26}$$

$$V^{m_1 m_2 m_3} = \frac{1}{3!} \epsilon^{m_1 \dots m_6} C_{m_4 m_5 m_6}, \quad U = \frac{1}{6!} \epsilon^{m_1 \dots m_6} C_{m_1 \dots m_6},$$

$$\varphi \propto \log \det (g_{\mu\nu})$$

Algebra of generalised vielbeins

$$\mathcal{L}_{E_A} E^M{}_B = E^m{}_A \partial_m E^M{}_B + \underbrace{\left(- \delta_N{}^m \delta_L{}^M + 10 d^{MmR} d_{NLR} \right)}_{\text{projector to ad of } E_6} \partial_m E^L{}_A E^N{}_B$$

$$= \mathcal{F}_{A,B}{}^C E^M{}_C, \tag{27}$$

Nambu-Lie duality

Initial data: fields in $E_A{}^M$, structure constants $\mathcal{F}_{AB}{}^C$

1. E_d transformation

$$T_A \to C_A{}^B T_B, \qquad \mathcal{F}_{AB}{}^C \to C_A{}^D C_B{}^E (C^{-1})_F{}^C \mathcal{F}_{DE}{}^F \\ E_A{}^M \to C_A{}^B E_B{}^M. \tag{28}$$

2. Condition: New generators form an EDA
3. New generalised frame $E_A{}^M$ defines new fields

$$g'_{mn}, \quad C'_{mnk}, \quad g'_{\mu\nu} = (\det g_{mn})^{-\frac{1}{9-d}} \mathfrak{g}_{\mu\nu}. \tag{29}$$

Examples

- All non-trivial examples of NAT dual bg's are based on coset manifolds
- SL(5) EDA does not seem to have non-trivial dual realisation [EtM(20)]
- All non-trivial examples of non-abelian U-dualities known so far are given for group manifolds [EtM, Sakatani(21)]

1. Uplifts of PL T-duality and small extensions
2. Dualities between M-theory and Type IIB theory
3. Generalized Yang-Baxter deformations

Examples

When $C_A{}^B$ is an outer automorphism of $\mathfrak{e}_{d(d)}$ an EDA is always mapped into an EDA [EtM(20)].

Example: SO(5,5) EDA. Start with $\mathfrak{g}$ given by

$$
\begin{aligned}
&\text{bas}\,\mathfrak{g} = \{T_1, T_2, T_3, T_4, T_5\},\\
&f_{23}{}^1 = 1\,, \quad f_{34}{}^5 = 1\,, \quad f_{24}{}^3 = c_0 \qquad (c_0 < 1)\,.
\end{aligned} \tag{30}
$$

Background metric in the parametrisation $g = e^{x\,T_1} e^{y\,T_2} e^{z\,T_3} e^{u\,T_4} e^{v\,T_5}$

$$
\begin{aligned}
g_{ij} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1-c_0^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & (1+c_0)\,y & 0 \\ 1 & 0 & (1+c_0)\,y & c_0\,(1+c_0)\,y^{\,2}+z^{\,2} & z \\ 0 & 0 & 0 & z & 1 \end{bmatrix},\\
g_{\mu\nu} &= \text{diag}(1,1,1,1,1,1)
\end{aligned} \tag{31}
$$

Examples

The dual background data.

Transformed structure constants:

$$f'_{12}{}^{5} = -1\,, \qquad f'_{1}{}^{235} = -1\,, \qquad f'_{3}{}^{245} = -c_0\,. \tag{32}$$

Background fields:

$$g_{ij} = \frac{1}{(1-c_0^2+x^2)^{\frac{2}{3}}} \begin{pmatrix} -c_0^2 z^2 - \frac{y^2}{4} & \frac{xy}{4} & -c_0 xz & -c_0^2+x^2+1 & -\frac{y}{2} \\ \frac{xy}{4} & 1-\frac{x^2}{4} & 0 & 0 & \frac{x}{2} \\ -c_0 xz & 0 & 1-c_0^2 & 0 & 0 \\ 1-c_0^2+x^2 & 0 & 0 & 0 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 0 & 0 & -1 \end{pmatrix},$$

$$C_3 = \frac{1}{1-c_0^2+x^2}\left(\frac{xy}{2}\,dx\wedge dy\wedge dz + x\,dy\wedge dz\wedge dv - c_0\,z\,dx\wedge dy\wedge dv\right)$$

$$g_{\mu\nu} = \frac{(1-c_0^2+x^2)^{\frac{1}{3}}}{(1-c_0^2)^{\frac{1}{4}}}\,\mathrm{diag}(1,1,1,1,1,1)\,.$$

Examples

The obtained background

- Is a solution of 11-dimensional supergravity equations of motion with wrong signature (M*-theory [Hull(98)])

- The duality is pure 11-dimensional and is not an uplift of Poisson-Lie T-duality.

- If identify $z \sim z + c^z$ the background is a U-fold. I.e. for the generalised metric $M_{MN} = E_M{}^A E_N{}^B M_{AB}$:

$$\begin{aligned} M(x,y,z+c^z) &= \Omega_{c^z} M(x,y,z) \Omega^T_{c^z}, \\ \Omega^z_c &= \exp\big(-c_0 c^z R_{245}\big) \in SO(5,5). \end{aligned} \tag{33}$$

Conclusions

- Exceptional geometry allow extension of Poisson-Lie T-dualities to symmetries of M-theory backgrounds: Nambu-Lie U-dualities
- The duality is defined in terms of Nambu-Poisson structures

$$\begin{aligned} 0 &= \beta^{k[m_1}\nabla_k\beta^{m_2m_3]}, \\ 0 &= \pi^{m_1m_2k}\,\nabla_n\pi^{n_1n_2l} - 3\,\pi^{k[n_1n_2}\,\nabla_k\pi^{l]m_1m_2}. \end{aligned} \tag{34}$$

- Several examples have been generated. All based on group manifolds, no rule useful for cosets is known.
- Generalised Yang-Baxter deformations are Nambu-Lie U-duality transformations. These are known for any backgrounds with at least three isometries [Bakhmatov, Gubarev, EtM(20), Gubarev, EtM(20)].

Future perspective

- Classification of exceptional Drinfeld algebras. (Something is already known [Hlavaty (20)])
- Non-abelian U-duality for coset spaces and for general backgrounds with isometries. More sensible examples.
- Generalisation of Drinfeld twist to EDA's: deformations of SCFT's dual to M-theory backgrounds.
- Nambu mechanics, integrability and all that.

Thank you for attention!