Coloured Poincaré algebra and corresponding particles

Karapet Mkrtchyan

Imperial College London

Integrability, Holography, Higher-Spin Gravity and Strings
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Sakharov Statue in Yerevan

Image from the Wikipedia page about Andrei Sakharov

Sakharov for peace: a concrete example

Sakharov on Artsakh (Karabakh)

"For Azerbaijan the issue of Karabakh is a matter of ambition, for the Armenians of Karabakh, it is a matter of life and death."

"If anyone was in doubt before Sumgait whether Nagorno-Karabakh should belong to Azerbaijan, then after this tragedy no one can have the moral right to insist that it should."

...

Stepanakert, October 2020. Image credit: AP

Reference

Based on:

Joaquim Gomis, Euihun Joung, Axel Kleinschmidt and KM
"Colourful Poincaré symmetry, gravity and particle actions"
(arXiv:2105.01686)

Motivation from Field Theory

The general problem

Four properties that are hard to combine in a classical field theory:

1. Extended (bosonic) space-time symmetries,
2. Local action principle,
3. Unitarity,
4. Non-trivial bulk propagation.

Examples are available with **any** three of these properties.
No satisfactory example is available with all four.

Motivation from Field Theory

The general problem

Four properties that are hard to combine in classical gravity:

1. Extended (bosonic) space-time symmetries,
2. Local action principle,
3. Unitarity,
4. Non-trivial bulk propagation.

Known examples with three of these properties

2,3,4: Einstein-Hilbert Gravity, SUGRA.
1,3,4: Vasiliev's Higher Spin Gravity.
1,2,4: Conformal Gravity (and much more).
1,2,3: Higher Spin (Chern-Simons) Gravities in $3d$.

Another Motivation

The general problem

Particle in a constant electromagnetic background can be described using Maxwell algebra (see, e.g., Gomis, Kleinschmidt '17):

$$[M_{ab}, M_{cd}] = \eta_{ac}\, M_{bd} + \dots\,, \qquad [M_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a\,,$$

$$[P_a, P_b] = Z_{ab}\,, \qquad [M_{ab}, Z_{cd}] = \eta_{ac} Z_{bd} + \dots\,, \qquad [Z_{ab}, Z_{cd}] = 0\,.$$

A natural question: what if we want to develop a theory of a particle in a constant Yang-Mills background? One needs to colour-decorate the Maxwell algebra, which, in turn requires Coloured Poincaré.

Colouring Isometry

Assume we have a theory of gravity perturbatively defined around a vacuum with isometry algebra $\mathfrak{g}_i$ (e.g., Poincaré). Coloured version of this algebra is a product of the isometry with a color factor:

$$\mathfrak{g} = \mathfrak{g}_i \otimes \mathfrak{g}_c$$

with elements:

$$F^I_X = M_X \otimes T^I \,, \quad M_X \in \mathfrak{g}_i \,, \quad T^I \in \mathfrak{g}_c$$

When is this product a Lie algebra? Commutator is given as:

$$[M_X \otimes T^I, M_Y \otimes T^J] = \frac{1}{2}\,[M_X, M_Y] \otimes \{T^I, T^J\} + \frac{1}{2}\,\{M_X, M_Y\} \otimes [T^I, T^J]$$

Coloured Isometry

In case if the algebra $\mathfrak{g}_i$ ($\mathfrak{g}_c$) is not associative, the last (first) term is not well defined unless the algebra $\mathfrak{g}_c$ ($\mathfrak{g}_i$) is commutative. For $\mathfrak{g}_i$ = Poincaré, the $\mathfrak{g}_c$ has to be commutative and associative. (Wald '87)

Adding physically relevant conditions:

- Positive-definite bilinear form (unitarity)
- Symmetric structure constants (derived from the cubic vertex)

Then, $\mathfrak{g}_c$ is a direct sum of one-dimensional algebras. Corresponding multi-gravity is described by a sum of Einstein-Hilbert actions with no cross interaction (Boulanger-Damour-Gualtieri-Henneaux '01).

Coloured Isometry

Commutator of the product algebra:

$$[M_X \otimes T^I, M_Y \otimes T^J] = \frac{1}{2}[M_X, M_Y] \otimes \{T^I, T^J\} + \frac{1}{2}\{M_X, M_Y\} \otimes [T^I, T^J]$$

if the isometry algebra $\mathfrak{g}_i$ is associative, the colour algebra $\mathfrak{g}_c$ does not have to be commutative: non-trivial Coloured isometry possible

Example: Coloured Higher-Spin algebra

No-go Theorems vs Yes-go examples

No-Go's

Interactions between multiple massless spin-two fields are trivial.
Wald '86-'87; Boulanger-Damour-Gualtieri-Henneaux '01

Yes-Go's

Vasiliev system allows for colour decoration (contains higher-spins, AdS background, flat limit not understood).
Konstein-Vasiliev '89

Coloured Gravity in $(A)dS_3$ (higher spins not necessary, but can be added)
Gwak, Joung, KM, Rey '15

Coloured $(A)dS_3$ Gravity

The AdS_3 isometry algebra is $\mathfrak{so}(2,2) \sim \mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R)$. One can extend it by two more generators to an associative algebra $\mathfrak{gl}(2,R) \oplus \mathfrak{gl}(2,R)$ which can be coloured. The two extra generators correspond to vector field (spin-one).

Key to colouring gravity in three dimensions: vector gauge fields.

Coloured AdS_3 algebra:

$$\mathfrak{g} = (\mathfrak{gl}_2 \oplus \mathfrak{gl}_2) \otimes \mathfrak{u}(N)$$

Coloured $(A)dS_3$ Gravity

Coloured AdS_3 algebra:

$$\mathfrak{g} = (\mathfrak{u}(1,1) \oplus \mathfrak{u}(1,1)) \otimes \mathfrak{u}(N) = \mathfrak{u}(N,N) \oplus \mathfrak{u}(N,N)$$

The action for Coloured Gravity can be given in a Chern-Simons form (similar to Achucaro-Townsend AdS_3 Chern-Simons gravity):

$$S = \frac{\kappa}{4\pi} \int_{M_3} Tr(\mathcal{A} \wedge \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})$$

There is still some ambiguity in the choice of bilinear form. We choose it uniquely by requiring the gravity to be Einstein-Hilbert.

Poincaré algebra in three dimensions

Poincaré algebra in $d = 3$ ($M_a = \frac{1}{2}\epsilon_{abc} M^{bc}$):

$$[M_a, M_b] = \epsilon_{ab}{}^c M_c\,, \qquad [M_a, P_b] = \epsilon_{ab}{}^c P_c\,, \qquad [P_a, P_b] = 0\,,$$

can be realised as:

$$M_a = L_a \otimes \mathcal{I}\,, \qquad P_a = i\, L_a \otimes \mathcal{J}\,,$$

where

$$\mathcal{I}^2 = \mathcal{I}\,, \qquad \mathcal{I}\,\mathcal{J} = \mathcal{J}\,\mathcal{I} = \mathcal{J}\,, \qquad \mathcal{J}^2 = 0\,.$$

and

$$[L_a, L_b] = \epsilon_{ab}{}^c L_c$$

$$L_a \in \mathfrak{su}(1,1) \sim \mathfrak{sl}(2,R) \sim \mathfrak{so}(1,2)$$

Associative extension of $3d$ Poincaré

Extending $\mathfrak{su}(1,1)$ to $\mathfrak{u}(1,1)$ adding a unit element,

$$L_a\, L_b = \frac{1}{2}\,\epsilon_{ab}{}^{c}\, L_c + \frac{1}{4}\,\eta_{ab}\,\mathbf{I}_2\,,$$

we can extend Poincaré to associative algebra:

$$\mathfrak{A} = \mathfrak{u}(1,1)\otimes a = \langle L_a\otimes\mathcal{I},\, L_a\otimes\mathcal{J},\, \mathbf{I}_2\otimes\mathcal{I}\,, \mathbf{I}_2\otimes\mathcal{J}\rangle\ ,$$

which can be multiplied by $\mathfrak{u}(N)$ colour algebra:

$$\mathrm{cPoin}_3 = \mathfrak{A}\otimes\mathfrak{u}(N)$$

Coloured Poincaré algebra

$$\mathrm{cPoin}_3 = \mathfrak{u}(N,N) \inplus_{\mathrm{adj}} \mathfrak{u}(N,N)$$

Coloured Poincaré generators

Coloured Lorentz:

$$M_a^{\widehat{I}} = \mathsf{i}\, L_a \otimes \mathcal{I} \otimes T^{\widehat{I}}\,, \qquad N^{\widehat{I}} = \mathbf{I}_2 \otimes \mathcal{I} \otimes T^{\widehat{I}}\,,$$

Coloured Translations:

$$P_a^{\widehat{I}} = L_a \otimes \mathcal{J} \otimes T^{\widehat{I}}\,, \qquad Q^{\widehat{I}} = \mathsf{i}\,\mathbf{I}_2 \otimes \mathcal{J} \otimes T^{\widehat{I}}\,.$$

Coloured Lorentz algebra $\mathfrak{u}(N,N)$

Generators

Coloured rotations: $M_0^{\widehat{I}} = (M_0\,,\, M_0^I)$,
Coloured boosts: $M_1^{\widehat{I}} = (M_1\,,\, M_1^I)\,,\; M_2^{\widehat{I}} = (M_2\,,\, M_2^I)$
Internal rotations: $N^{\widehat{I}} = (N\,,\, N^I)$
N is the central element

Commutators

$$\left[M_a^{\widehat{I}}, M_b^{\widehat{J}}\right] = \frac{1}{2}\,\varepsilon_{ab}{}^{c}\,\widehat{d}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,M_c^{\widehat{K}} - \frac{1}{4}\,\eta_{ab}\,\widehat{f}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,N^{\widehat{K}}\,,$$

$$\left[M_a^{\widehat{I}}, N^{\widehat{J}}\right] = \widehat{f}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,M_a^{\widehat{K}}\,, \qquad \left[N^{\widehat{I}}, N^{\widehat{J}}\right] = \widehat{f}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,N^{\widehat{K}}$$

Coloured Translations

Translations

Spacetime: $P_a^{\widehat{I}} = (P_a\,,\,P_a^I)$, internal: $Q^{\widehat{I}} = (Q\,,\,Q^I)$

Commutators

Lorentz with Translations:

$$\left[M_a^{\widehat{I}}, P_b^{\widehat{J}}\right] = \frac{1}{2}\,\varepsilon_{ab}{}^c \widehat{d}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,P_c^{\widehat{K}} - \frac{1}{4}\,\eta_{ab}\,\widehat{f}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,Q^{\widehat{K}}\,,$$

$$\left[M_a^{\widehat{I}}, Q^{\widehat{J}}\right] = \widehat{f}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,P_a^{\widehat{K}}\,,$$

$$\left[N^{\widehat{I}}, P_a^{\widehat{J}}\right] = \widehat{f}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,P_a^{\widehat{K}}\,, \qquad \left[N^{\widehat{I}}, Q^{\widehat{J}}\right] = \widehat{f}^{\,\widehat{I}\widehat{J}}{}_{\widehat{K}}\,Q^{\widehat{K}}\,,$$

Translations:

$$\left[P_a^{\widehat{I}}, P_b^{\widehat{J}}\right] = 0\,, \qquad \left[P_a^{\widehat{I}}, Q^{\widehat{J}}\right] = 0\,, \qquad \left[Q^{\widehat{I}}, Q^{\widehat{J}}\right] = 0\,.$$

Coloured Minkowski Gravity

Remove the center of the Coloured Poincaré (N, Q)

$$M_a^{\widehat{I}} = (M_a\,, M_a^I)\,, \quad N^I\,, \quad P_a^{\widehat{I}} = (P_a\,, P_a^I)\,, \quad Q^I$$

$$(M_a\,,\, M_a^I\,,\, N^I) \in \mathfrak{su}(N, N)$$

Action

$$S = \frac{\kappa}{4\pi} \int_{M_3} Tr(\mathcal{A} \wedge \mathcal{A} + \frac{2}{3}\, \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})\,,$$

$$\mathcal{A} \in \mathfrak{su}(N, N) \in_{\text{adj}} \mathfrak{su}(N, N)$$

The bilinear form uniquely fixed by requiring the gravity to be Einstein-Hilbert.

Massive Coloured Particle action – nonlinear realisation

Reference momentum

$$P_a = m\,\delta_a^0\,, \qquad P_a^I = 0\,, \qquad Q^I = 0\,.$$

The little group

The stability group of this momentum is span by:

$$(M_0\,, \quad M_0^I\,, \quad N^I) \in \mathfrak{u}(1) \oplus \mathfrak{su}(N) \oplus \mathfrak{su}(N)$$

Massive Coloured Particle action – nonlinear realisation

Reference momentum

$$\widehat{\mathbb{P}} = m \begin{pmatrix} \mathrm{i}\mathbf{I} & 0 \\ 0 & -\mathrm{i}\mathbf{I} \end{pmatrix}$$

Boost elements

The elements of the factor group

$$G/H = SU(N,N)/(U(1) \otimes SU(N) \otimes SU(N))$$

are of the form:

$$b = \exp\left[\frac{1}{2}\begin{pmatrix} 0 & \mathbf{V}^{+} \\ \mathbf{V}^{-} & 0 \end{pmatrix}\right], \quad \text{with} \quad (\mathbf{V}^{+})^{\dagger} = \mathbf{V}^{-}\,.$$

Massive Coloured Particle action – nonlinear realisation

Reference momentum

$$\widehat{\mathbb{P}} = m \begin{pmatrix} \mathrm{i}\mathbf{I} & 0 \\ 0 & -\mathrm{i}\mathbf{I} \end{pmatrix}$$

The orbit of the reference momentum

$$b\,\widehat{\mathbb{P}}\,b^{-1} = m \begin{pmatrix} \mathrm{i}\cosh\sqrt{\mathbf{V}^{+}\mathbf{V}^{-}} & -\mathrm{i}\frac{\sinh\sqrt{\mathbf{V}^{+}\mathbf{V}^{-}}}{\sqrt{\mathbf{V}^{+}\mathbf{V}^{-}}}\mathbf{V}^{+} \\ \mathrm{i}\mathbf{V}^{-}\frac{\sinh\sqrt{\mathbf{V}^{+}\mathbf{V}^{-}}}{\sqrt{\mathbf{V}^{+}\mathbf{V}^{-}}} & -\mathrm{i}\cosh\sqrt{\mathbf{V}^{-}\mathbf{V}^{+}} \end{pmatrix}$$

Covariant equation

$$\mathbb{P}^2 + m^2\,\mathbb{I} = 0$$

Coloured Particle

Lagrangian

$$L = \frac{1}{2N} \mathrm{Tr} \left[\dot{\mathbb{X}}\, \mathbb{P} + \mathbb{L} \left(\mathbb{P}^2 + m^2 \mathbb{I} \right) \right]$$

Semisimple Orbits

The solutions of the covariant equation $\mathbb{P}^2 + m^2\mathbb{I} = 0$ are parameterized by diagonal matrices (up to $\mathfrak{su}(N,N)$ rotations)

$$\widehat{\mathbb{P}}_\ell = m\, \mathrm{diag}(\underbrace{i, \ldots, i}_{\ell}, \underbrace{-i, \ldots, -i}_{N-\ell}, \underbrace{i, \ldots i}_{N-\ell}, \underbrace{-i, \ldots, -i}_{\ell}),$$

There are $N+1$ orbits: $\ell = 0, 1, \ldots, N$

For $N = 1$, two orbits: positive and negative energy particles.

Coloured Particle variations

Massless coloured particle action

$$S = \frac{1}{N} \int dt \, \mathrm{Tr} \left[\mathbb{P} \dot{\mathbb{X}} + \mathbb{L} \, \mathbb{P}^2 \right]$$

AdS Coloured Particle action

$$S = \frac{1}{N} \int dt \, \mathrm{Tr} \left[\mathbb{P} \dot{\mathbb{X}} + \mathbb{L} \left(((1 - \mathbb{X})\mathbb{P}(1 + \mathbb{X}))^2 + m^2 \, \mathbb{I} \right) \right]$$

Coloured Minkowski space

Coloured Minkowski point

$$\mathbb{X} = x^a\, P_a + x_I^a\, P_a^I + y_I\, Q^I\ ,$$

Coloured Poincaré transformations

Lie algbra element $\Lambda = \mathbb{O} + \mathbb{A}$ with

$$\mathbb{O} = \underbrace{\omega^a\, M_a + \omega_I^a\, M_a^I + \sigma_I\, N^I}_{\text{coloured Lorentz}}, \qquad \mathbb{A} = \underbrace{\alpha^a\, P_a + \alpha_I^a\, P_a^I + \beta_I\, Q^I}_{\text{coloured translations}}$$

$$\delta_\Lambda \mathbb{X} = [\mathbb{O}, \mathbb{X}] + \mathbb{A}$$

Coloured Minkowski space

Coloured Minkowski point

$$\mathbb{X} = x^a \, P_a + x_I^a \, P_a^I + y_I \, Q^I \,,$$

Coloured Poincaré transformations

$$\begin{aligned} \delta_\Lambda x^a &= \varepsilon_{bc}{}^a \, \omega^b \, x^c + \tfrac{1}{N} \, \delta^{IJ} \, \varepsilon_{bc}{}^a \, \omega_I^b \, x_J^c + \alpha^a \,, \\ \delta_\Lambda x_I^a &= \varepsilon_{bc}{}^a \, \omega^b x_I^c + \varepsilon_{bc}{}^a \, \omega_I^b \, x^c + \tfrac{1}{2} \, \varepsilon_{bc}{}^a \, d^{JK}{}_I \, \omega_J^b \, x_K^c \\ &\quad + f^{JK}{}_I \, \omega_J^a \, y_K - f^{JK}{}_I \, \sigma_J \, x_K^a + \alpha_I^a \,, \\ \delta_\Lambda y_I &= -\tfrac{1}{4} \, \eta_{ab} \, f^{JK}{}_I \, \omega_J^a \, x_K^b - f^{JK}{}_I \, \sigma_J y_K + \beta_I \,. \end{aligned}$$

Distance in Coloured Minkowski

A metric on coloured Minkowski space that is invariant under coloured Lorentz transformations is given by

$$\frac{1}{2N}\mathrm{Tr}(\mathbb{X}^2) = x^a x^b \eta_{ab} + \frac{1}{N} x_I^a x_J^b \eta_{ab} \delta^{IJ} - \frac{1}{N} y_I y_J \delta^{IJ} \,,$$

where η_{ab} is the $(-++)$ Minkowski metric and δ^{IJ} the $SU(N)$ invariant metric.

Coloured Particle action

Integral of the line element — "geometric action"

$$S_{\text{geo}} = m \int d\tau \sqrt{-\mathrm{Tr}\,[\dot{\mathbb{X}}^2]}$$

In the Hamiltonian form:

$$S_{geo} = \frac{1}{2N} \int d\tau \, \mathrm{Tr} \left[\dot{\mathbb{X}}\,\mathbb{P} + e \left(\mathbb{P}^2 + m^2 \mathbb{I} \right) \right]$$

Compare to the NLR action

$$S_{NLR} = \frac{1}{2N} \int d\tau \, \mathrm{Tr} \left[\dot{\mathbb{X}}\,\mathbb{P} + \mathbb{L} \left(\mathbb{P}^2 + m^2 \mathbb{I} \right) \right]$$

Outlook

What next?

- Coloured Maxwell algebra, particle in Yang-Mills background.
- Field theory of Coloured Particles — CG with matter.
- Extended objects?
- Higher dimensions?

Thank you for your attention!