Anomalies and Supersymmetry

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Anomalous global symmetry
(flavor, R-symmetry, higher form, higher spin)

$\downarrow$ Supersymmetry

Noether current multiplet
(couples to vector/gravity multiplet)

“Small multiplet”

- N=1 vector multiplet in WZ gauge
- Conformal current multiplet
- R-multiplet
- Extended supersymmetry (N>1)
- ...

$\downarrow$

Deformation of the
supercharge algebra

“Large multiplet”

- N=1 vector multiplet in superspace
- Ferrara-Zumino multiplet
- S-multiplet (?)
- ...

References

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Outline

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Noether's theorem

Continuous global symmetries* lead to a conserved current

$$\partial_\mu \mathcal{J}^\mu = 0$$

and an associated conserved charge

$$\mathcal{Q} = \int_{\mathcal{C}_t} d^{d-1}x \; \mathcal{J}^t$$

The conserved charges satisfy an algebra

$$\big[\mathcal{Q}_I, \mathcal{Q}_J\big\} = f_{IJ}{}^K \mathcal{Q}_K, \qquad I, J, K = 1, 2, 3, \cdots$$

* I will focus exclusively on symmetries connected to the identity

(Perturbative) Anomalies

Quantum anomalies arise due to the breaking of classical symmetries by the UV regulator or the path integral measure

They are manifest in Ward identities and the charge algebra

$$\mathcal{D}_\mu \langle \mathcal{J}^\mu \rangle = -\mathcal{A}$$

$$\partial_\mu \langle \mathcal{J}^\mu(x_1) \mathcal{J}^\nu(x_2) \mathcal{J}^\rho(x_3) \rangle = \text{contact terms}$$

$$\big[\mathcal{Q}_I, \mathcal{Q}_J \big\} = f_{IJ}{}^K \mathcal{Q}_K + k_{IJ}$$

The structure of anomalies is determined by the Wess-Zumino (WZ) consistency conditions up to overall numerical coefficients

The algebra deformation k_{IJ} is determined by the anomaly $\mathcal{A}$

Background fields

Global symmetries can be studied by turning on background gauge fields A_μ that couple to the Noether currents $\mathcal{J}^\mu$, i.e. (to linear order)

$$Z[A] = e^{iW[A]} = \int \mathcal{D}\phi \exp\Big(iS[\phi] + i\int d^dx\; A\cdot\mathcal{J}\Big)$$

The full, non-linear, functional $W[A]$ is obtained by gauging the global symmetry and integrating over the matter fields only – not A_μ

$W[A]$ is the generating function of connected current correlators

$$\langle \mathcal{J}^{\mu_1}(x_1)\mathcal{J}^{\mu_2}(x_2)\cdots\mathcal{J}^{\mu_n}(x_n)\rangle = \frac{\delta^n W[A]}{\delta A_{\mu_1}(x_1)\delta A_{\mu_2}(x_2)\cdots\delta A_{\mu_n}(x_n)}$$

't Hooft anomalies

If $Z[A]$ transforms with a phase under a gauge transformation $A_\mu \to A + \mathcal{D}\vartheta$ and the phase cannot be removed by a local counterterm the theory has a 't Hooft anomaly

$$Z[A + \mathcal{D}\vartheta] = Z[A] \exp\left(i \int d^d x \, \vartheta \cdot \mathcal{A}(A)\right)$$

For infinitesimal gauge parameter $\vartheta(x)$ this is equivalent to

$$\delta_\vartheta W[A] = \underbrace{G(\vartheta, A)}_{\text{integrated anomaly}} = \int d^d x \, \vartheta \cdot \underbrace{\mathcal{A}(A)}_{\text{local anomaly}}$$

The gauge dependence of the effective action is equivalent to the non conservation of the current

$$\delta_\vartheta W[A] = \int d^d x \, \underbrace{\frac{\delta W}{\delta A^a_\mu}}_{\langle \mathcal{J}^\mu_a \rangle} (\mathcal{D}_\mu \vartheta)^a = \int d^d x \, \vartheta \cdot \mathcal{A}(A) \;\; \Rightarrow \;\; \mathcal{D}_\mu \langle \mathcal{J}^\mu \rangle = -\mathcal{A}(A)$$

Wess-Zumino conditions

The anomalies must satisfy the Wess-Zumino (WZ) consistency conditions, whose infinitesimal form is

$$(\delta_{\vartheta_1}\delta_{\vartheta_2} - \delta_{\vartheta_2}\delta_{\vartheta_1})W[A] = \delta_{[\vartheta_1,\vartheta_2]}W[A]$$

or

$$\delta_{\vartheta_1}G(\vartheta_2, A) - \delta_{\vartheta_2}G(\vartheta_1, A) = G([\vartheta_1, \vartheta_2], A)$$

Any ***local*** functional $W_{\rm loc}[A]$ is a trivial solution of the WZ conditions

Nontrivial solutions of the WZ conditions are local functionals

$$G(\vartheta, A) = \int d^d x\, \vartheta \cdot \mathcal{A}(A)$$

that are not the gauge variation of a local functional $W_{\rm loc}[A]$

BRST transformations

Solving the WZ conditions is a cohomology problem

This is formalized by replacing the infinitesimal gauge parameter ϑ^a with a Grassmann-valued Faddeev-Popov ghost v^a so that

$$\begin{aligned} \delta_\vartheta A &= \mathrm{d}A + [A,\vartheta] = \mathcal{D}\vartheta \\ \delta_\vartheta F &= [F,\vartheta] \end{aligned} \quad \to \quad \begin{aligned} sA &= -\,\mathrm{d}v - Av - vA = -\mathcal{D}v \\ sF &= Fv - vF \\ sv &= -\,\tfrac{1}{2}[v,v] = -v^2 \end{aligned}$$

where $F = \mathrm{d}A + A^2$ is the field strength of A

The BRST operator s is nilpotent and so the WZ condition becomes

$$s^2 W[A] = 0 \quad \Leftrightarrow \quad sG(v,A) = 0$$

Nontrivial ***consistent*** anomalies are elements of the cohomology of s

BRST algebra

The cohomology problem that determines the consistent anomaly can be solved using the anomaly descent procedure that follows from the BRST algebra [Stora '77, '84; Zumino '84; Mañes, Stora, Zumino '85]

$$\mathrm{d}^2 = 0, \qquad s^2 = 0, \qquad (\mathrm{d}+s)^2 = 0$$

We start by defining the quantities

$$\widehat{A} \equiv A + v, \qquad \widehat{F} \equiv (\mathrm{d}+s)\widehat{A} + \widehat{A}^2$$

The nilpotency of d and $\mathrm{d}+s$ imply respectively the Bianchi identities

$$\mathrm{d}F + AF - FA = 0, \qquad (\mathrm{d}+s)\widehat{F} + \widehat{A}\widehat{F} - \widehat{F}\widehat{A} = 0$$

These imply that the corresponding anomaly polynomials are closed

$$\mathrm{d}P_{d+2}(F) = 0, \qquad (\mathrm{d}+s)P_{d+2}(\widehat{F}) = 0$$

Russian formula and anomaly descent

The next key ingredient is the "Russian formula"

$$\widehat{F} = F$$

Together with the above results, this implies that

$$\begin{aligned} 0 &= P_{d+2}(\widehat{F}) - P_{d+2}(F) \\ &= (\mathrm{d} + s)\Omega_{d+1}(\widehat{A}, \widehat{F}) - \mathrm{d}\Omega_{d+1}(A, F) \\ &= s\Omega^{(0)}_{d+1}(A, F) + \mathrm{d}\Omega^{(1)}_{d}(v, A, F) + \mathcal{O}(v^2) \end{aligned}$$

where $\Omega^{(k)}_{d+1-k}$ determine the expansion of the Chern-Simons form

$$\Omega_{d+1}(\widehat{A}, \widehat{F}) = \Omega_{d+1}(\widehat{A}, F) = \sum_{k \geq 0} \Omega^{(k)}_{d+1-k}(v, A, F)$$

Grouping terms of equal ghost number gives the ***descent equations***

$$s\Omega^{(k)}_{d+1-k}(v, A, F) + \mathrm{d}\Omega^{(k+1)}_{d-k}(v, A, F) = 0$$

Consistent anomaly

The solution to the BRST cohomology problem is obtained from the $k = 1$ descent relation, which integrated over M_d, with $\partial M_d = \emptyset$, gives

$$s \int_{M_d} \Omega_d^{(1)}(v, A, F) = - \int_{M_d} \mathrm{d}\Omega_{d-1}^{(2)}(v, A, F) = 0$$

We identify the ***consistent anomaly***

$$G(v, A) = N \int_{M_d} \Omega_d^{(1)}(v, A, F)$$

where N is a normalization factor determined by a 1-loop calculation

$$N = \frac{i^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} \left(\frac{d+2}{2}\right)!}(n_L - n_R), \quad d \text{ even}, \quad n_{L,R} = \text{\# of chiral fermions}$$

Anomaly inflow

The descent relation for $k = 0$ leads to a toy version of ***anomaly inflow***. Integrating over M_{d+1} with $\partial M_{d+1} = M_d$ we obtain

$$s \int_{M_{d+1}} N\Omega^{(0)}_{d+1}(A, F) = -N \int_{M_d} \Omega^{(1)}_d(v, A, F) = -G(v, A)$$

Hence, the Chern-Simons form $\Omega^{(0)}_{d+1}(A, F)$ “cancels” the anomaly

$$s\Big(W[A] + N \int_{M_{d+1}} \Omega^{(0)}_{d+1}(A, F)\Big) = 0$$

This relation provides a constructive way to determine the consistent anomaly, given the general expression for the Chern-Simons form

$$\Omega^{(0)}_{d+1}(A, F) = \Omega_{d+1}(A, F) = \tfrac{d+2}{2} \int_0^1 dt\, P_{d+2}\Big(A, F_t^{\frac{d}{2}}\Big)$$

where $F_t \equiv tF + (t^2 - t)A^2$

BZ polynomial and the covariant current

The anomaly $G(\vartheta, A)$ implies that the consistent current $\mathcal{J}^\mu$ does not transform covariantly under gauge transformations

$$\delta_\vartheta \langle \mathcal{J}_a^\mu \rangle = \delta_\vartheta \Big(\frac{\delta W[A]}{\delta A_\mu^a} \Big) = \delta_\vartheta \Big(\frac{\delta}{\delta A_\mu^a} \Big) W[A] + \frac{\delta G(\vartheta, A)}{\delta A_\mu^a}$$
$$= [\langle \mathcal{J}^\mu \rangle, \vartheta]_a + \frac{\delta G(\vartheta, A)}{\delta A_\mu^a}$$

Bardeen and Zumino (BZ) [Bardeen, Zumino '84] showed that there exists a local polynomial $\mathcal{X}^\mu(A)$ that transforms as

$$\delta_\vartheta \mathcal{X}_a^\mu(A) = [\mathcal{X}^\mu(A), \vartheta]_a - \frac{\delta G(\vartheta, A)}{\delta A_\mu^a}$$

It follows that the sum of $\mathcal{J}^\mu$ and $\mathcal{X}^\mu(A)$ does transform covariantly

$$\langle \mathcal{J}_{\rm cov}^\mu \rangle \equiv \langle \mathcal{J}^\mu \rangle + \mathcal{X}^\mu(A), \qquad \delta_\vartheta \langle \mathcal{J}_{\rm cov}^\mu \rangle = [\langle \mathcal{J}_{\rm cov}^\mu \rangle, \vartheta]$$

Covariant anomaly

The BZ polynomial is also determined by the anomaly polynomial

$$\int d^dx\,\delta A\cdot\mathcal{X}(A) = -N^{\frac{d(d+2)}{4}}\int_0^1 dt\,t\int_{M_d} P_{d+2}\Big(\delta A, A, F_t^{\frac{d-2}{2}}\Big)$$

where again $F_t \equiv tF + (t^2-t)A^2$ and δA is a general variation of A

The BZ polynomial, and hence the covariant current, cannot be expressed as the variation of an effective action in M_d

The divergence of the covariant current is

$$\mathcal{D}_\mu\langle\mathcal{J}^\mu_{\rm cov}\rangle = -\mathcal{A}_{\rm cov}(A)$$

where the ***covariant anomaly***

$$G_{\rm cov}(v,A) = \int_{M_d} d^dx\,v\cdot\mathcal{A}_{\rm cov}(A) = N^{\frac{d+2}{2}}\int_{M_d} P_{d+2}\Big(v, F^{\frac{d}{2}}\Big)$$

depends only on the field strength F

An identity

There is a general local relation between the BZ polynomial and the consistent anomaly that plays an important role in the following

$$F^a_{\mu\nu}\mathcal{X}^\nu_a(A) - A^a_\mu \mathcal{A}_a(A) = 0$$

or equivalently, for any vector field κ^μ,

$$\Omega^{(1)}_d(i_k A, A, F) + \frac{d(d+2)}{4}\int_0^1 dt\, t P_{d+2}\big(i_k F, A, F_t^{\frac{d-2}{2}}\big) = 0$$

e.g. for the Abelian case in $d = 4$

$$F_{\tau\nu}\mathcal{X}^\nu(A) - A_\tau \mathcal{A}(A) \propto \epsilon^{\mu\nu\rho\sigma}\big(F_{\mu\nu}F_{\rho\sigma}A_\tau + 4A_\nu F_{\rho\sigma}F_{\tau\mu}\big) = 0$$

since antisymmetrizing five indices in four dimensions gives zero

$$F_{[\mu\nu}F_{\rho\sigma}A_{\tau]} = 0$$

Outline

Ward identities

The anomalous conservation of the consistent current in the presence of an arbitrary gauge field A_μ is an example of a ***Ward identity***

$$\mathcal{D}_\mu \langle \mathcal{J}^\mu \rangle = -\mathcal{A}(A) \quad (*)$$

As we saw, this is obtained by writing $\delta_\vartheta W[A]$ in two different ways

$$\delta_\vartheta W[A] = \int d^d x \, \vartheta \cdot \mathcal{A}(A) = \int d^d x \, \underbrace{\frac{\delta W}{\delta A^a_\mu}}_{\langle \mathcal{J}^\mu \rangle} \underbrace{(\mathcal{D}_\mu \vartheta)^a}_{\delta_\vartheta A^a_\mu}$$

where the first equality is a consequence of the WZ conditions, while the second follows from the chain rule and the definition of $\langle \mathcal{J}^\mu \rangle$

Successive derivatives of $(*)$ with respect to A_μ lead to the Ward identities for any correlator of consistent currents

$$\partial_{\mu_1} \langle \mathcal{J}^{\mu_1}(x_1) \mathcal{J}^{\mu_2}(x_2) \cdots \mathcal{J}^{\mu_n}(x_n) \rangle = \text{contact terms}$$

Spacetime symmetries

Spacetime symmetries, such as diffeomorphisms and local Lorentz transformations, act on multiple background fields

As a result, the Ward identities for these symmetries contain several operators, besides the energy-momentum tensor

As a warmup for supersymmetry, let us consider the diffeomorphism Ward identity in the presence of an anomalous flavor symmetry

The current operators related to these symmetries are

$$\langle \mathcal{T}_a^\mu \rangle = e^{-1} \frac{\delta W}{\delta e_\mu^a}, \qquad \langle \mathcal{J}^\mu \rangle = e^{-1} \frac{\delta W}{\delta A_\mu}, \qquad e \equiv \det(e_\mu^a)$$

(Recall that $g_{\mu\nu} = e_\mu^a \, e_\nu^b \, \eta_{ab}$)

Gravitational anomalies

The transformations of $W[e, A]$ under diffeomorphisms and Lorentz transformations define the diffeomorphism and Lorentz anomalies

$$\delta_\xi W[e, A] = G_D(\xi, \Gamma, R, F), \qquad \delta_\lambda W[e, A] = G_L(\lambda, \omega, R, F)$$

where Γ and ω are respectively the Christoffel and spin connections

G_L can be determined by standard $SO(d)$ descent, while G_D may be obtained from an ***effective*** $GL(d)$ descent with parameter $\partial_\mu \xi^\nu$

In fact, G_L and G_D are related by a local counterterm and so one of the two can always be set to zero [Bardeen, Zumino '84]

A counterterm shifts also the mixed gauge-gravitational anomaly between diffeomorphisms and gauge transformations and so both G_L and G_D can be set to zero unless $d = 4k + 2$, $k = 0, 1, \ldots$

Diffeomorphism Ward identity

The diffeomorphism Ward identity follows from the transformation law

$$\delta_\xi W[e,A] = \int d^d x\, e\, \xi \cdot \mathcal{A}_D = \int d^d x\, e \big(\delta_\xi e^a_\mu \langle \mathcal{T}^\mu_a \rangle + \delta_\xi A_\mu \cdot \langle \mathcal{J}^\mu \rangle\big)$$

$$= -\int d^d x\, e\, \xi^\mu \big(e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle - F_{\mu\nu} \cdot \langle \mathcal{J}^\nu \rangle + A_\mu \cdot \mathcal{D}_\nu \langle \mathcal{J}^\nu \rangle - {\omega_\mu}^{ab} e_{\nu[a} \langle \mathcal{T}^\nu_{b]} \rangle\big)$$

The terms involving the flavor current can be simplified as

$$-F_{\mu\nu} \cdot \langle \mathcal{J}^\nu \rangle + A_\mu \cdot \mathcal{D}_\nu \langle \mathcal{J}^\nu \rangle = -F_{\mu\nu} \cdot \langle \mathcal{J}^\nu_{\mathrm{cov}} \rangle + \underbrace{F_{\mu\nu} \cdot \mathcal{X}^\nu(A) - A_\mu \cdot \mathcal{A}(A)}_{=0}$$

Hence, only the flavor field strength and covariant current appear

$$\boxed{e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle - F_{\mu\nu} \cdot \langle \mathcal{J}^\nu_{\mathrm{cov}} \rangle - {\omega_\mu}^{ab} e_{\nu[a} \langle \mathcal{T}^\nu_{b]} \rangle = -\mathcal{A}_{D\mu}}$$

This extends to the gravitational anomaly and stress tensor, i.e. $\mathcal{T} \to \mathcal{T}_{\mathrm{cov}}$, $\mathcal{A}_D \to \mathcal{A}_D^{\mathrm{cov}}$ using the gravitational BZ polynomial

Ward identity covariance

The appearance of the covariant current in the diffeomorphism Ward identity can be understood using the WZ consistency conditions

Besides the diffeo-diffeo and gauge-gauge WZ conditions, $W[e, A]$ satisfies the mixed diffeo-gauge condition

$$(\delta_\vartheta \delta_\xi - \delta_\xi \delta_\vartheta) W = \delta_{\vartheta'} W, \qquad \vartheta' = \mathcal{L}_\xi \vartheta$$

However, $G_D(\xi, \Gamma, R, F)$ does not depend on A and $G(\vartheta, A, F, R)$ does not depend on Γ and so the following identities hold separately

$$\delta_\vartheta \delta_\xi W = \delta_\vartheta G_D(\xi, \Gamma, R, F) = 0$$

$$\delta_\xi \delta_\vartheta W + \delta_{\vartheta'} W = \delta_\xi G(\vartheta, A, F, R) + G(\vartheta', A, F, R) = 0$$

The covariance of the diffeomorphism Ward identity follows immediately from the first of these, namely $\delta_\vartheta \delta_\xi W = 0$

Supersymmetry Ward identity

In supersymmetric theories W satisfies the WZ condition

$$(\delta_\vartheta \delta_\epsilon - \delta_\epsilon \delta_\vartheta) W = 0$$

However, using the general form of the flavor anomaly we determine

$$\delta_\epsilon \delta_\vartheta W = \delta_\epsilon G(\vartheta, A) = -\delta_\vartheta \int d^d x \, \delta_\epsilon A \cdot \mathcal{X}[A] \neq 0$$

Hence, in contrast to $\delta_\vartheta \delta_\xi W = 0$ for diffeos, $\delta_\vartheta \delta_\epsilon W \neq 0$. In particular,

$$\delta_\epsilon W = -\int d^d x \, \underbrace{\delta_\epsilon A \cdot \mathcal{X}[A]}_{\text{universal}} + \underbrace{\text{gauge invariant}}_{\text{multiplet dependent}} \neq 0$$

The gauge invariant part is determined by the $\delta_\epsilon \delta_{\epsilon'}$ WZ condition

The supersymmetry Ward identity, therefore, takes the general form

$$\int d^d x \, (-\bar{\epsilon} \, \partial_\mu \langle \mathcal{S}^\mu \rangle + \delta_\epsilon A_\mu \cdot \langle \mathcal{J}^\mu_{\text{cov}} \rangle) = \text{gauge invariant}$$

Example: 4d $\mathcal{N} = 1$ vector multiplet

As a first example to illustrate this structure, let us consider 4d $\mathcal{N} = 1$ theories with an anomalous Abelian flavor symmetry

$$\partial_\mu \langle \mathcal{J}^\mu \rangle = \frac{\kappa}{4} \widetilde{F}^{\mu\nu} F_{\mu\nu}, \qquad \widetilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

A_μ and $\mathcal{J}^\mu$ belong to vector multiplets, which in the WZ gauge comprise respectively $(A_\mu, \lambda^\alpha, D)$ and $(\mathcal{J}^\mu, \mathcal{O}^\lambda_\alpha, \mathcal{O}_D)$

The flavor current and background multiplets are related as usual by

$$\langle \mathcal{J}^\mu \rangle = \frac{\delta W}{\delta A_\mu}, \qquad \langle \mathcal{O}^\lambda_\alpha \rangle = \frac{\delta W}{\delta \lambda^\alpha}, \qquad \langle \mathcal{O}_D \rangle = \frac{\delta W}{\delta D}$$

Supersymmetry Ward identity

The WZ conditions take the form

$$(\delta_\vartheta\delta_{\vartheta'}-\delta_{\vartheta'}\delta_\vartheta)W=0,\qquad (\delta_\varepsilon\delta_\vartheta-\delta_\vartheta\delta_\varepsilon)W=0$$

$$(\delta_\varepsilon\delta_{\varepsilon'}-\delta_{\varepsilon'}\delta_\varepsilon)W=(\delta_\xi+\delta_\vartheta)W,\quad \xi^\mu=-2i(\varepsilon\sigma^\mu\bar\varepsilon-\varepsilon'\sigma^\mu\bar\varepsilon),\quad \vartheta=-\xi^\mu A_\mu$$

Their solution is [Itoyama, Nair, Ren ’85; Guadagnini, Mintchev ’86]

$$\delta_\vartheta W=-\frac{\kappa}{4}\int d^4x\,\vartheta\,\widetilde F^{\mu\nu}F_{\mu\nu},\qquad \kappa=\text{constant}$$

$$\delta_\varepsilon W=\int d^4x\,\Big(-\underbrace{i\varepsilon\sigma_\mu\bar\lambda}_{\delta_\varepsilon A_\mu}\,\underbrace{\kappa\,\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}}_{\mathcal{X}^\mu(A)}+\underbrace{3\kappa i\,\varepsilon\lambda\bar\lambda^2}_{\text{gauge invariant}}+h.c.\Big)\neq 0$$

This leads to the supersymmetry Ward identity

$$\partial_\mu\langle\mathcal{S}^\mu_\alpha\rangle+i(\sigma_\mu\bar\lambda)_\alpha\langle\mathcal{J}^\mu_{\text{cov}}\rangle+\big(iD\delta_\alpha{}^\beta+\tfrac{1}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta F_{\mu\nu}\big)\langle\mathcal{O}^\lambda_\beta\rangle$$
$$-(\sigma^\mu\partial_\mu\bar\lambda)_\alpha\langle\mathcal{O}_D\rangle=3\kappa i\lambda_\alpha\bar\lambda^2$$

Outline

Generalized anomaly descent

The structure of anomalies in supersymmetric theories can be determined by solving the full set of WZ consistency conditions

However, systematic methods for solving these, such as the anomaly descent, must be generalized to accommodate supersymmetry

For minimal rigid supersymmetry in dimensions 2 to 6 (off-shell) and up to 10 (on-shell), versions of a supersymmetric anomaly descent have been discussed in [Itoyama, Nair, Ren '85; Guadagnini, Mintchev '86, Kaiser '88; Altevogt, Kaiser '88; Baulieu, Martin '08]

Our discussion is based on the BRST version of the supersymmetric anomaly descent developed in [Kaiser '88; Altevogt, Kaiser '88], which is based on the usual, purely bosonic, anomaly polynomial $P_{d+2}(F)$

BRST and supersymmetry

In a supersymmetric anomaly descent one must be careful with the fermions across dimensions, but since no fermions are present in $P_{d+2}(F)$ we need only consider fermions in d+1 and d dimensions

The structure of the BRST algebra is generic, but let us focus on 3d $\mathcal{N}=1$ rigid supersymmetry, which corresponds to 2d $\mathcal{N}=(1,1)$

$$\delta_Q(\epsilon)A_\mu = -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda, \qquad \delta_Q(\epsilon)\lambda_\mu = \tfrac{1}{4}(\gamma^{\rho\sigma}\epsilon)_\mu F_{\rho\sigma}$$

Supersymmetry requires a second BRST operator, c, such that

$$\begin{aligned} sA_\mu &= \mathcal{D}_\mu v & cA_\mu &= -\tfrac{1}{2}\bar{\alpha}\gamma_\mu\lambda^a + a^\nu\partial_\nu A_\mu \\ s\lambda &= \{\lambda, v\} & c\lambda &= \tfrac{1}{4}\gamma^{\mu\nu}F^a_{\mu\nu}\alpha + a^\nu\partial_\nu\lambda \\ sv &= -v^2 & cv &= \tfrac{1}{4}\bar{\alpha}\gamma^\mu\alpha A_\mu + a^\nu\partial_\nu v \\ s\alpha &= 0 & ca^\mu &= -\tfrac{1}{4}\bar{\alpha}\gamma^\mu\alpha \end{aligned}$$

where (v^a, α, a^μ) are Faddeev-Popov ghosts

BRST algebra

The above BRST transformations satisfy the algebra

$$\mathrm{d}^2 = 0, \quad s^2 = 0, \quad (s+c)^2 = 0, \quad (\mathrm{d}+s)^2 = 0, \quad (\mathrm{d}+s+c)^2 = 0$$

Also for diffeos where $\mathcal{L}_\xi \to s_{GL(d)} + c$. But diffeos satisfy also $c^2 = 0$, which facilitates a reformulation in terms of a standard $GL(d)$ descent

The WZ conditions are equivalent to the BRST cohomology problem

$$(s+c)^2 W[A,\lambda] = 0 \;\; \Leftrightarrow \;\; (s+c)G(\{v,\alpha,a\},A,\lambda) = 0$$

In addition to the quantities $\widehat{A} \equiv A + v$, $\widehat{F} \equiv (\mathrm{d}+s)\widehat{A} + \widehat{A}^2$ we define

$$\widehat{\mathcal{A}} \equiv \widehat{A} + u = A + v + u, \qquad \widehat{\mathcal{F}} \equiv (\mathrm{d}+s+c)\widehat{\mathcal{A}} + \widehat{\mathcal{A}}^2$$

where u is unspecified (will set $u = 0$ later)

The nilpotency of $\mathrm{d}+s+c$ implies the Bianchi identity

$$(\mathrm{d}+s+c)\widehat{\mathcal{F}} + \widehat{\mathcal{A}}\widehat{\mathcal{F}} - \widehat{\mathcal{F}}\widehat{\mathcal{A}} = 0$$

Generalized descent equations

Recall that a key ingredient of the standard descent is the “Russian formula” $\widehat{F} = F$, which implies that $P_{d+2}(\widehat{F}) - P_{d+2}(F) = 0$

However, $\widehat{\mathcal{F}} \neq F$ for any choice of u and so we define

$$P_{d+2}(\widehat{\mathcal{F}}) - P_{d+2}(F) = (\mathrm{d}+s+c)\Omega_{d+1}(\widehat{\mathcal{A}}, \widehat{\mathcal{F}}) - \mathrm{d}\Omega_{d+1}(A, F) \equiv \sum_{k\geq 1} X^{(k)}_{d+2-k}$$

Expanding in total ghost number gives ***two*** sets of descent equations

$$\boxed{(s+c)X^{(k+1)}_{d+1-k} + \mathrm{d}X^{(k+2)}_{d-k} = 0, \quad X^{(k+1)}_{d+1-k} = (s+c)\Omega^{(k)}_{d+1-k} + \mathrm{d}\Omega^{(k+1)}_{d-k}}$$

where $\Omega^{(k)}_{d+1-k}$ determine the expansion of the Chern-Simons form

$$\Omega_{d+1}(\widehat{A}, \widehat{G}) = \sum_{k\geq 0} \Omega^{(k)}_{d+1-k}(\{v, \alpha, a\}, A, F)$$

Supersymmetrized consistent anomaly

At $k=1$ the second set of descent equations gives

$$X_d^{(2)} = (s+c)\Omega_d^{(1)} + \mathrm{d}\Omega_{d-1}^{(2)}$$

while from the definition of $X_{d+1-k}^{(k+1)}$ follows that (with $u=0$)

$$X_d^{(2)} = \tfrac{d+2}{2} P_{d+2}\big(cv, F^{\frac{d}{2}}\big) + \tfrac{d(d+2)}{4} P_{d+2}\big(cA, cA, F^{\frac{d-2}{2}}\big)$$

Using this expression one finds that $X_d^{(2)}$ can be expressed as

$$X_d^{(2)} = (s+c)Y_d^{(1)} + \mathrm{d}Z_{d-1}^{(2)}$$

where $Y_d^{(1)} \neq \Omega_d^{(1)}$ is gauge invariant, i.e. $sY_d^{(1)} = 0$, and depends only on the c ghosts, α and a

It follows that the supersymmetrized consistent anomaly is

$$(s+c)\int_{M_d} \big(\Omega_d^{(1)} - Y_d^{(1)}\big) = 0 \;\Rightarrow\; G(\{v,\alpha,a\}, A, \lambda) = N\int_{M_d} \big(\Omega_d^{(1)} - Y_d^{(1)}\big)$$

Supersymmetric Chern-Simons

We have seen that the anomaly follows from the fact that $X_d^{(2)}$ can be written in "normal form", i.e. $X_d^{(2)} = (s+c)\Omega_d^{(1)} + \mathrm{d}\Omega_{d-1}^{(2)}$, $Y_d^{(1)} \neq \Omega_d^{(1)}$

However, a supersymmetric Chern-Simons form exists iff also

$$X_{d+1}^{(1)} = (s+c)Y_{d+1}^{(0)} + \mathrm{d}Z_d^{(1)}, \qquad Y_{d+1}^{(0)} \neq \Omega_{d+1}^{(0)}$$

In that case, the descent equations determine that

$$\begin{aligned} &(s+c)\big(\Omega_{d+1}^{(0)} - Y_{d+1}^{(0)}\big) = \mathrm{d}\big(Z_d^{(1)} - \Omega_d^{(1)}\big) \\ &\Omega_d^{(1)} - Z_d^{(1)} = \Omega_d^{(1)} - Y_d^{(1)} + (s+c)\text{-exact} \end{aligned}$$

which allows us to identify $\Omega_{d+1}^{(0)} - Y_{d+1}^{(0)}$ with a supersymmetric Chern-Simons form and $\Omega_d^{(1)} - Z_d^{(1)}$ with the anomaly

Example

From the definition of $X_{d+1-k}^{(k+1)}$ follows that

$$X_{d+1}^{(1)} = \tfrac{d+2}{2} P_{d+2}\big(cA, F^{\frac{d}{2}}\big)$$

The BRST transformations of the 3d $\mathcal{N} = 1$ vector multiplet determine

$$\begin{aligned} X_3^{(1)} &= \mathrm{tr}\,(-\bar{\alpha}\gamma\lambda F) + \mathrm{d}\,\mathrm{tr}\,(2 i_a A F) \\ \mathbf{c}\,\mathrm{tr}\,(\bar{\lambda}\lambda * \mathbb{1}) &= -\mathrm{tr}\,(\bar{\alpha}\gamma\lambda F) - \mathrm{d}\,\mathrm{tr}\,(*a\bar{\lambda}\lambda) \end{aligned}$$

Hence,

$$X_3^{(1)} = (\mathbf{s} + \mathbf{c}) \underbrace{\mathrm{tr}\,(\bar{\lambda}\lambda * \mathbb{1})}_{Y_3^{(0)}} + \mathrm{d}\, \underbrace{\mathrm{tr}\,(2 i_a A F - *a\bar{\lambda}\lambda)}_{Z_3^{(1)}}$$

and the 3d $\mathcal{N} = 1$ supersymmetric Chern-Simons (CS) form is

$$\boxed{\Omega_{\mathrm{CS}} = \Omega_3^{(0)} - Y_3^{(0)} = \mathrm{tr}\left(A\mathrm{d}A + \frac{2}{3}A^3 - \bar{\lambda}\lambda * \mathbb{1}\right)}$$

Outline

Supersymmetric anomaly inflow

Turning the argument around, supersymmetric CS forms can be used as a powerful tool to determine certain supersymmetrized anomalies

$$\delta_{\{\vartheta,\epsilon,\cdots\}} W = G(\{\vartheta,\epsilon,\cdots\},\{A,\omega,\cdots\}) = \delta_{\{\vartheta,\epsilon,\cdots\}} S_{\mathrm{CS}}$$

where

$$S_{\mathrm{CS}} = \int_{M_{d+1}} \Omega_{\mathrm{CS}}, \qquad M_d = \partial M_{d+1}$$

Placing S_{CS} on the LHS, i.e.

$$\delta_{\{\vartheta,\epsilon,\cdots\}}(W - S_{\mathrm{CS}}) = 0$$

gives an example of a codimension-1 supersymmetric anomaly inflow

Higher codimension supersymmetric inflows are realized e.g. on the worldvolume of D-branes and membranes

Example: 2d $\mathcal{N} = (p, q)$ flavor anomalies

The maximal vector multiplet with a pure CS action in 3d is the $\mathcal{N} = 3$ multiplet $(A_\mu, \lambda^I, \chi, \sigma^I, D^I)$, $I = 1, 2, 3$

The $\mathcal{N} = 3$ CS action is [Kao, Lee, Lee '95]

$$\mathcal{L}_{\rm CS} = \frac{k}{4\pi} \operatorname{tr} \Big(\varepsilon^{\mu\nu\rho} \big(A_\mu \partial_\nu A_\rho - \tfrac{2i}{3} A_\mu A_\nu A_\rho \big) \\ - \bar{\lambda}^I \lambda_I + \bar{\chi}\chi - 2\sigma^I D_I + \tfrac{i}{3} \varepsilon_{IJK} \sigma^I [\sigma^J, \sigma^K] \Big)$$

Varying this action determines the anomalies for the 2d $\mathcal{N} = (p, q)$, $p, q \leq 3$, vector multiplet obtained by dimensional reduction

$$\delta_G(\vartheta) W = \frac{k}{4\pi} \varepsilon^{\hat{\nu}\hat{\rho}} \int d^2x \operatorname{tr} (\vartheta \, \partial_{\hat{\nu}} A_{\hat{\rho}})$$

$$\delta_Q(\epsilon) W = \frac{k}{4\pi} \int d^2x \operatorname{tr} \big(\delta_Q(\epsilon) A_{\hat{\nu}} \underbrace{\varepsilon^{\hat{\nu}\hat{\rho}} A_{\hat{\rho}}}_{\mathcal{X}(A)} - \underbrace{2\,\sigma^I (\varepsilon_{IJK} \bar{\epsilon}^J \gamma_* \lambda^K + \bar{\epsilon}_I \gamma_* \chi)}_{\text{gauge invariant}} \big)$$

Outline

Summary

- In several commonly encountered Noether current multiplets of anomalous global symmetries supersymmetry is violated

$$\delta_\epsilon W = -\int d^dx \, \underbrace{\delta_\epsilon A \cdot \mathcal{X}[A]}_{\text{universal}} + \underbrace{\text{gauge invariant}}_{\text{multiplet dependent}} \neq 0$$

- The supersymmetry violating terms are determined by the WZ consistency conditions in the presence of supersymmetry
- In many cases, these can be solved by a generalized anomaly descent procedure, or a supersymmetric Chern-Simons form
- The supersymmetry violating terms affect the dependence of (refined) supersymmetric observables on bosonic fugacities
- Anomalous higher form and higher spin symmetries?

Thank you for your attention!