Gribov Ambiguity and Degenerate Systems

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Outline

Gribov Ambiguity

- Generating functional for Yang-Mills Theory

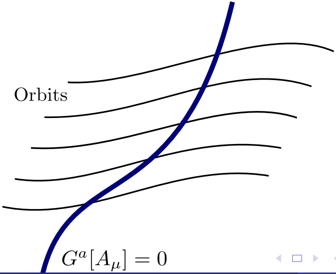
$$Z = \int DA\, e^{-iS}$$

- Action

$$S = -\frac{1}{4}\int d^4x\,\mathrm{tr}\left[F^{\mu\nu}F_{\mu\nu}\right]$$

where $F^a_{\mu\nu}$ the field stregh associated to $A_\mu = A^a_\mu T_a$

- To avoid overcounting we must fix the gauge $G^a\left[A_\mu\right] = 0$

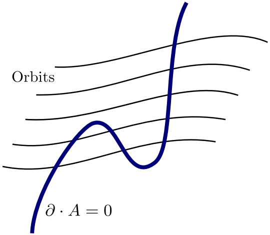


Gribov Ambiguity

- The restriction is carried out using the Fadeev-Popov method

$$Z = \int DA\,\delta\left(G^a\left[A_\mu\right]\right)\det\mathcal{M}\,e^{iS} \quad , \quad \mathcal{M}^a{}_b(x,y) = \frac{\delta G^a\left[A^g_\mu(x)\right]}{\delta\alpha^b(y)}$$

- Coulomb gauge does not fix the gauge completely $\Longrightarrow$ Gribov copies **[Gribov (1978)]**



- Same for all gauge fixing conditions **[Singer(1978)]**.

Gribov Ambiguity

- The condition for this to happen is

$$G^a\left[g^{-1}A_\mu g + g^{-1}\partial_\mu g\right] = 0 \quad , \quad g \neq \mathbf{1}$$

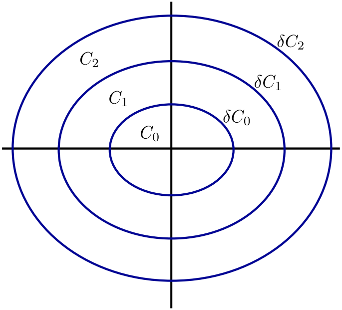
- Infinitesimal gauge transformations, $\delta A_\mu = D_\mu \alpha$

$$G^a\left[\left(A_\mu + D_\mu \alpha\right)\right] = 0$$

$$\Longrightarrow \int d^4y \mathcal{M}^a{}_b(x,y)\alpha^b(y) = 0$$

- Infinitesimal Gribov copies $\rightarrow$ zero modes of the Faddeev-Popov operator
- The functional integral Z is ill-defined

Gribov Ambiguity



- Gribov proposed to restrict the path integral to the Gribov region

$$C_0 \equiv \left\{ A_\mu , G^a[A_\mu] = 0 \middle| \det \mathcal{M} > 0 \right\}$$

- C_0 is bounded and convex **[van Baal (1992)]**
- All orbits intersect the Gribov region **[Dell'Antonio, Zwanziger (1991)]**

Gribov Ambiguity

- The restriction can be implemented in the form

$$Z_G = \mathcal{N} \int DA\delta\left(\partial^\mu A_\mu\right) \det\left(\mathcal{M}\right) \exp\left(-S_{YM}\right) \mathcal{V}\left(C_0\right)$$

- The factor $\mathcal{V}\left(C_0\right)$ ensures integration only over C_0.
- Gluon propagator is modified: $D^{ab}_{\mu\nu}\left(q\right) = \delta^{ab} g_0^2 \frac{q^2}{q^4+\gamma^4}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$. **[Gribov (1978)]**
- Imaginary poles $\rightarrow$ gluons are not in the spectrum $\rightarrow$ Confinement
- Studies at finite temperature show a critical T for which imaginary poles disappear **[Canfora, Pais, Salgado-Rebolledo (2014)]**
- Restriction to the Gribov horizon can be properly implemented to match with lattice results **[Sorella et al (2008)]**

Degenerate Systems

- Hamiltonian Systems $\longrightarrow$ Symplectic geometry
- Symplectic manifold $=$ (M, Ω)

$$\Omega = d\mathcal{A}$$

- First order action

$$L = \mathcal{A}_A \dot{z}^A - H$$

- Poisson Bracket $=$ Inverse of Ω

$$\{z^A, z^B\} = \Omega^{AB}$$

- Euler-Lagrange equations

$$\Omega_{AB} \dot{z}^A = \partial_B H$$

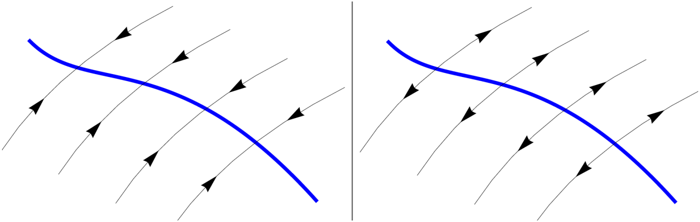
- $\det \Omega \neq 0 \Longrightarrow$ Regular systems

Degenerate Systems

- $\det \Omega = 0$ with fixed rank $\Longrightarrow$ Local Symmetries
- $\det \Omega = 0$ and non-constant rank $\Longrightarrow$ Degenerate systems.

$$\Omega_{AB} \dot{z}^A = \partial_B H$$

- Degeneracy surfaces $\Sigma = \{z \in \Gamma / \det \Omega = 0\}$



- Divide phase space into dynamical disconnected regions [**Saavedra, Troncoso, Zanelli (2001)**]
- The measure for the Hilbert space vanishes at the degeneracy surfaces [**de Michelli, Zanelli (2012)**]

Gribov Ambiguity as Degeneracy

- Consider a system with a finite number of degrees of freedom and a local symmetry.

$$\begin{aligned} S &= \int dt \; L(x) \\ \delta S &= 0 \;\; \text{for some } \delta x \end{aligned}$$

- Local symmetry $\rightarrow$ constraints.
- In the Hamiltonian formalism there are primary constraints

$$\varphi_m(x) \approx 0$$

- Dirac Formalism: Preservation in time of these can lead to secondary constraints, tertiary constraints, etc

Gribov Ambiguity as Degeneracy

- They can be classified in first and second class

$$\varphi_M = (\phi_i, \gamma_\alpha)$$

- First class constraints = generators of the local symmetries
- Second class constraints can be eliminated by implementing Dirac brackets

$$\{F, G\}^* = \{F, G\} - \{F, \gamma_\alpha\} C^{\alpha\beta} \{\gamma_\beta, G\}$$

where

$$C_{\alpha\beta} = \{\gamma_\alpha, \gamma_\beta\}$$

- Quantization $\rightarrow$ fix the gauge $\rightarrow$ extra constraints G_i such that first class constraints become second class.

$$\gamma_I = (\phi_i, G_j)$$

- Defining Dirac brackets we can set all the constraints to zero strongly

Gribov Ambiguity as Degeneracy

- Proper gauge fixing:
  1. Accessibility
  2. Complete gauge fixation [**Henneaux, Teitelboim (1992)**]
- Dirac brackets $\longrightarrow$ Symplectic structure of the reduced phase space.

$$\{y^a, y^b\}^* = \Omega_{red}^{ab}$$

$$\Omega_{red} = \frac{1}{2}\Omega_{ab}^{red} dy^a \wedge dy^b$$

- We can redefine the Dirac matrix by defining $\gamma_I \rightarrow \bar{\gamma}_I = V_{IJ}\gamma_J$

$$\bar{C} = V^T C V = \begin{pmatrix} & & & & & 1 \\ & & & & \cdots & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdots & & & & \\ -1 & & & & & \end{pmatrix}$$

Gribov Ambiguity as Degeneracy

- In other words we use new coordinates $z^A = (\bar{\gamma}_I, y^a)$
- Implementing the constraints strongly, the path integral in Hamiltonian form is

$$Z = \int Dy\ e^{iS} = \int Dz \prod_I \delta\left(\bar{\gamma}_I\right)\ e^{iS}$$

- Turning back to the old variables

$$Z = N \int Dx \prod_I \delta\left(\gamma_I\right) \det\{G_i, \phi_j\}\, e^{iS}$$

- $\det\{G_i, \phi_j\}$ is identified with the Faddeev-Popov determinant and

$$\mathcal{M}_{ij} = \{G_i, \phi_j\}$$

- If the system has Gribov ambiguity then

$$\det\{G_i, \phi_j\} = 0 \text{ at the Gribov horizon}$$

Gribov Ambiguity as Degeneracy

- Dirac matrix

$$C_{IJ} = \{\gamma_I, \gamma_J\} = \begin{pmatrix} \{G_i, G_j\} & \mathcal{M}_{ij} \\ -\mathcal{M}_{ij} & \{\phi_i, \phi_j\} \end{pmatrix} .$$

- Therefore $\det C \approx (\det \mathcal{M})^2$
- In the new coordinates

$$\{z^A, z^B\} = \begin{pmatrix} \{y^a, y^b\} & 0 \\ 0 & C_{IJ} \end{pmatrix}$$

$$\det \Omega^{-1} = \det \Omega_{red}^{-1} (\det \mathcal{M})^2$$

- Ω regular $\implies \det \Omega_{red}^{-1}$ blows up at the Gribov horizon

$$\implies \det \Omega_{red} = 0 \text{ at the Gribov horizon}$$

- **Theorem**: In the presence of Gribov ambiguity the reduced system is degenerate **[Canfora, de Michelli, Salgado-Rebolledo, Zanelli (2015)]**.

FLPR Model

- Solvable model [**Friedberg, Lee, Pang, Ren (1995)**].

$$L=\frac{1}{2}\left((\dot{x}+\alpha yq)^2+(\dot{y}-\alpha xq)^2+(\dot{z}-q)^2\right)-V(\rho)$$

- Canonical momenta

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x}+\alpha yq\,, \quad & p_y &= \frac{\partial L}{\partial \dot{y}} = \dot{y}-\alpha xq\,, \\ p_z &= \frac{\partial L}{\partial \dot{z}} = \dot{z}-q\,, \quad & p_q &= \frac{\partial L}{\partial \dot{q}} = 0 \end{aligned}$$

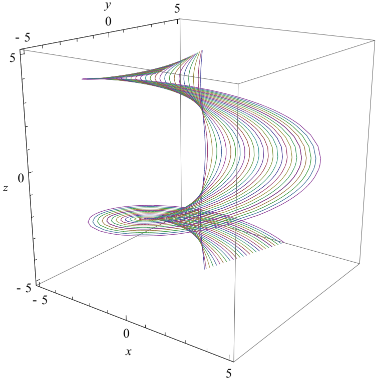
- First class constraints

$$\varphi = p_q \approx 0$$

$$\phi = p_z + \alpha\left(xp_y - yp_x\right) \approx 0$$

FLPR Model

- ϕ generates helicoidal orbits $\delta_\phi(x, y, z, q) = \epsilon(t)(-\alpha y, \alpha x, 1, 0)$



- Gauge condition

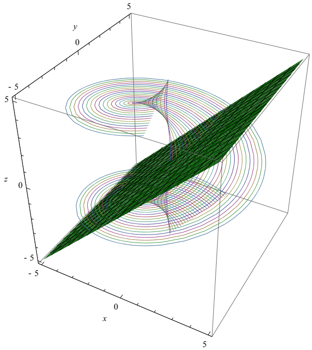
$$G = z - \lambda x \approx 0$$

- G presents Gribov Ambiguity $\mathcal{M} = \{G, \phi\} = 1 + \alpha\lambda y$

FLPR Model

- The pair G , ϕ is second class everywhere, except at the Gribov horizon

$$\Xi = \{(x, p_x, y, p_y, z, p_z) \in \Gamma \,|\, \mathcal{M} = 0\}$$



FLPR Model

- Second class constraints $\{G, \phi\}$

$$\gamma_I :\ \gamma_1 = G = z - \lambda x\ , \qquad \gamma_2 = \phi = p_z + \alpha(xp_y - yp_x)$$

- Setting constraints strongly equal to zero $\rightarrow$ z and p_z eliminated from phase space
- Dirac matrix

$$C_{IJ} = \begin{pmatrix} 0 & \mathcal{M} \\ -\mathcal{M} & 0 \end{pmatrix}$$

- Dirac brackets

$$[x, p_x]^* = \frac{1}{\mathcal{M}}\ ,\ \ [x, y]^* = 0\ ,\ \ [x, p_y]^* = 0\ ,$$
$$[y, p_y]^* = 1\ ,\ \ [y, p_x]^* = \frac{\alpha\lambda x}{\mathcal{M}}\ ,\ \ [p_x, p_y]^* = -\frac{\alpha\lambda p_x}{\mathcal{M}}$$

FLPR Model

- Reduced symplectic form is

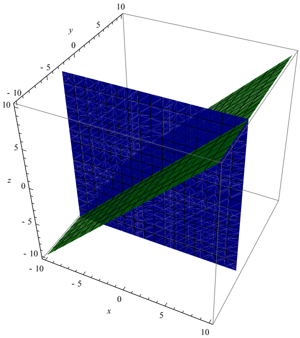
$$\omega_{ab} = \begin{pmatrix} 0 & -\mathcal{M} & -\alpha\lambda p_x & \alpha\lambda x \\ \mathcal{M} & 0 & 0 & 0 \\ \alpha\lambda p_x & 0 & 0 & -1 \\ -\alpha\lambda x & 0 & 1 & 0 \end{pmatrix} .$$

- Closed but degenerates precisely at the Gribov horizon

$$\det[\omega_{ab}] = \mathcal{M}^2$$

$$\Sigma = \{(x, p_x, y, p_y) \in \Gamma_0 | \Upsilon(u) \equiv \mathcal{M} = 0\}$$

FLPR Model



- The degeneracy surface divides phase space into dynamically disconnected regions

$$C_+ := \{(x,y,z)\,|\, z - \lambda x = 0,\, 1 + \alpha\lambda y > 0\}\ ,$$
$$C_- := \{(x,y,z)\,|\, z - \lambda x = 0,\, 1 + \alpha\lambda y < 0\}\ .$$

Conclusions

- We have studied Gribov ambiguity from a Hamiltonian point of view
- It has been shown that, for finite dimensional systems, the presence of Gribov copies implies a degeneracy for the reduced phase space
- We have studied the FLPR model and found the degenerate reduced symplectic form in the presence of a Gribov horizon
- The degeneracy surface divides phase space into dynamically disconnected regions
- This suggests that the restriction to the Gribov horizon in QCD is natural

Future Directions

- To look for explicit degeneracies in the symplectic form for Yang-Mills theories after gauge fixing [WORK IN PROGRESS]
- In Yang-Mills theory the canonical momenta associated to the gauge fied A_μ^a is

$$\Pi_a^\mu = \frac{\partial \mathcal{L}}{\partial \left(\dot{A}_\mu^a\right)} = F_a^{\mu 0}.$$

- There is a primary constraint

$$\phi_a^0 = \Pi_a^0 \approx 0$$

- The canonical hamiltonian is given by

$$H = \int d^3x \left(\dot{A}_i^a \Pi_a^i - \mathcal{L}\right) = \int d^3x \left(\mathcal{H}_0 + A_0^a \left(D_i\right)_a{}^b \Pi_b^i\right)$$

where

$$\mathcal{H}_0 = \frac{1}{2}\Pi_a^i \Pi_i^a + \frac{1}{4} F_{ij}^a F_a^{ij}$$

Future Directions

- Total hamiltonian

$$H_T = H + \int d^3x \mu^a \phi_a^0$$

- Preservation in time of the primary constraint leads to

$$\phi_a = -\left(D_i\right)_a{}^b \Pi_b^i \approx 0$$

- The set $\left\{\phi_a^0, \phi_a\right\}$ is first class.
- Eliminating ϕ_a^0 and A_0^a the extended action

$$S_E = \int dx^0 \int d^3x \left(\dot{A}_a^i \Pi_i^a - \mathcal{H}_0 - \lambda^a \phi_a\right)$$

is invariant only under the transformations generated by ϕ_a

$$\delta A_i^a\left(x\right) = \int d^3y \epsilon^b\left(y\right)\left\{A_i^a\left(x\right), \phi_b\left(y\right)\right\} = \left(D_i\right)_a{}^b \epsilon^b\left(x\right)$$

Future Directions

- The first class constraints satisfy $\{\phi_a, \phi_b\} = f^c{}_{ab}\phi_c$
- To fix the gauge we choose the Coulomb condition $G^a = \partial^i A_i^a \approx 0$
- Now the set $\gamma_A = \left(\phi_a, G^b\right)$ is second class
- Dirac matrix

$$C_{AB}\left(x,y\right) = \begin{pmatrix} 0 & -\partial^i \left(D_i\right)^a{}_b\, \delta^3\left(x-y\right) \\ \partial^i \left(D_i\right)^a{}_b\, \delta^3\left(x-y\right)_b & 0 \end{pmatrix}$$

- Eigenvalue equation

$$-\partial^i \left(\delta^a{}_b \partial_i + i f^a_{cb} A^c_i\right) \alpha^b = \epsilon\left(A_i\right) \alpha^a \ .$$

Future Directions

- For vanishing gauge potentials $-\partial^i\partial_i\alpha^a = \epsilon\alpha^a$ has positive eigenvalues $\epsilon = p^2$
- For small enough gauge fields A_i^a there are only positive eigenvalues
- For sufficiently large gauge fields, a zero mode $\epsilon = 0$ can appear
- This will be a zero mode of the Dirac Matrix and of the reduced symplectic form
- Set the constraints strongly to zero and evaluate Dirac brackets
- Compute the reduced phase space symplectic form and look for degeneracies
- Generalization for the theory at finite temperature
- In the finite temperature case the degeneracy should disappear at some critical temperature

Thank You !