No-π Theorem for Euclidean Massless Correlators

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Starting point: 1991

The seminal calculation /Gorishnii, Kataev, Larin/ of the $\mathcal{O}(\alpha_s^3)$ Adler function demonstrated for the first time a mysterious complete cancellation of **all** contributions proportional to ζ_4 (abounding in separate diagrams) while odd zetas ζ_3 and ζ_5 survive! The result is π-**free** ($\zeta_4 = \frac{\pi^4}{90}$ and $\zeta_6 = \frac{\pi^6}{945}$)

$$d_2 = -\frac{3}{32}C_F^2 + C_F T_f \left[\zeta_3 - \frac{11}{8}\right] + C_F C_A \left[\frac{123}{32} - \frac{11\zeta_3}{4}\right],$$

$$d_3 = -\frac{69}{128}C_F^3 + C_F^2 T_f \left[-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right] + C_F T_f^2 \left[\frac{151}{54} - \frac{19}{9}\zeta_3\right] + C_F^2 C_A \left[-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right]$$
$$+ C_F T_f C_A \left[-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_F C_A^2 \left[\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right],$$

the authors wrote: **"We would like to stress the cancellations of ζ_4 in the final results for $R(s)$. It is very interesting to find the origin of the cancellation of ζ_4 in the physical quantity."**

The situation got even more interesting about 20 years later: the $\mathcal{O}({\alpha_s}^4)$ contributions to the Adler function and to the coefficient function (CF) of C_{Bjp} the Bjorken sum rule /Baikov, Kühn, K. Ch. (2009-2010)/ were found to be

completely π-free★

★ **we do not consider any powers of π which are routinely generated during the procedure of analytical continuation to the Minkowskian (negative) values of the momentum transfer Q^2)**

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\,\zeta_3 + \frac{5}{3}\,\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\,\zeta_3 + \frac{125}{6}\,\zeta_5 + 3\,\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\,\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\,\zeta_3 - \frac{170}{9}\,\zeta_5 - \frac{1}{2}\,\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\,\zeta_3 - \frac{5}{12}\,\zeta_5 + \frac{1}{6}\,\zeta_3^2 EQN$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\,\zeta_3 - \frac{125}{4}\,\zeta_5 + \frac{105}{4}\,\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\,\zeta_3 + \frac{145}{24}\,\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\,\zeta_3 - \frac{5155}{48}\,\zeta_5 - \frac{33}{4}\,\zeta_3^2 - \frac{105}{8}\,\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\,\zeta_3 - \frac{95}{144}\,\zeta_5 - \frac{35}{4}\,\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\,\zeta_3 + \frac{18805}{288}\,\zeta_5 - \frac{11}{2}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\,\zeta_3 + \frac{1855}{288}\,\zeta_5 - \frac{11}{12}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\,\zeta_3 + \frac{2255}{32}\,\zeta_5 - \frac{1155}{16}\,\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\,\zeta_3 - \frac{1045}{48}\,\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\,\zeta_3 + \frac{6505}{48}\,\zeta_5 + \frac{1155}{32}\,\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\,\zeta_3 + \frac{55}{9}\,\zeta_5 + \frac{385}{16}\,\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{77995}{1152}\,\zeta_5 + \frac{605}{32}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{12545}{1152}\,\zeta_5 + \frac{121}{96}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$

Transcedentals: odd zetas: $\zeta_3, \zeta_5, \zeta_7$ BUT NOT even ones ζ_4 or ζ_6

What is common between the Adler function and C_{Bjp}? They both are "physical" (no anomalous dimension, depend only on the *bare* cc α_s).
The Adler function D^{SS} for the scalar correlator is π-dependent already at $\mathcal{O}({\alpha_s}^3)$★ and even more at the next loop (expilict ζ_4 and ζ_6 terms)★★

In fact, one can construct a physical (read: scale-independent) object from $\mathcal{O}(\alpha_s^L)$ D^{SS} and the (L+1)-loop quark mass anomalous dimension γ_m.

For $\mathcal{O}(\alpha_s^3)$ D^{SS} it was done with expected result: all π dependence indeed disappeared! /Vermaseren, Larin van Ritbergen (1997)/

BUT for $\mathcal{O}(\alpha_s^4)$ correlators this stopped to work:

It was found /Baikov, K. Ch. Kühn (2017)/ that ζ_4 does not dissappear from a scale-independent (SI) object constructed from $\mathcal{O}(\alpha_s^4)$ D^{SS} and 5-loop AD γ_m.

ζ_4 also does not dissappear from the 5-loop gluon correlator (enters the hadronic decays of the Higgs boson) computed in
/ Herzog, Ruijl, Ueda, Vermaseren and Vogt (2017)/.

★ K. K. Ch. (1997).

★★ Baikov, Kühn, K. Ch. (2006)

2017: 2 new important developments

- 5-loop QCD β-function and quark AD γ_m were computed /Baikov, K. Ch. Kühn; Herzog, Ruijl, Ueda, Vermaseren and Vogt; Luthe, Maier, Marquard and Schroder/.

  $\boxed{\text{First appearence of } \pi \text{ in } \beta_5 \text{ (in form of } \zeta_4)}$

- Jamin and Miravitllas have discovered that after a transition to a new so-called C-scheme all terms proportional to even zetas (ζ_4 and ζ_6) do disappear from (SI versions of) the 5-loop scalar correlator and the 5-loop gluon correlator (both enter the hadronic decays of the Higgs boson /Baikov, K. Ch. Kühn (2005); Herzog, Ruijl, Ueda, Vermaseren and Vogt (2017)).

  They also suggested that the absence of even zetas after transition to the C-scheme is an universal feature of *all* $\mathcal{O}(\alpha_s^5)$ physical quantities $\equiv$ $\boxed{\textbf{no } \pi\textbf{-conjecture}}$

  Later many more particular confirmations of the conjecture have been found and discussed and used for non-tivial check of many multiloop (4 and 5) results for ADs in /Davies and Vogt (2017); K. Ch, G. Falcioni, Herzog and Vermaseren (2017)/

A word about notations and conventions (goodbye β_0 and γ_0)

we use

1. $$\gamma(a) = \sum_{i\geq 1} \gamma_i \, a^i, \quad a = \frac{\alpha_s}{4\pi}$$

2. $$\beta(a) = \sum_{i\geq 1} \beta_i \, a^i$$

3. Landau gauge for QCD (for simplicity, could be relaxed)

4. G-scheme instead of $\overline{\text{MS}}$ one: all ADs and betas are *not* different from their $\overline{\text{MS}}$ versions but the simplest 1-loop p-integral is just identically equal $\frac{1}{\epsilon}$:

$$\frac{1}{i(2\pi)^D} \int \frac{d^D l}{(-l^2)(-(q-l)^2)} = \frac{1}{(4\pi)^2 \, (-q^2)^\epsilon} \frac{1}{\epsilon}$$

for finite renormalized quantities: $\left(\ln \frac{\mu^2}{Q^2}\right)_G \to \left(\ln \frac{\mu^2}{Q^2}\right)_{\overline{\text{MS}}} + 2$

2017: BIG PUZZLE

What is special in the C-scheme★?

$$a = \bar{a}\,(1 + c_1\,\bar{a} + c_2\,\bar{a}^2 + c_3\,\bar{a}^3 + c_4\bar{a}^4)$$

with c_1, c_2 and c_3 are made from $\beta_1 - \beta_4$ (all free from even zetas) and with

$$c_4 = \frac{1}{3}\frac{\beta_5}{\beta_1}$$

any SI $\mathcal{O}(\alpha_s^5)$ correlator $F(\bar{a})$ as well the very β-function $\boldsymbol{\bar{\beta}(\bar{a})}$ loose any dependence on even zetas. We will call the class of renormalization schemes for which

$$\boldsymbol{\bar{\beta}(\bar{a}) \stackrel{\pi}{=} 0}$$

as $\boldsymbol{\pi}$-*independent* schemes

★C-scheme has some interesting features and applications, not relevant in our context of π-hunting; see /Boito, Jamin and Miravitllas, [1606.06175]/

To really appreciate the mystery behind these cancellations induced by the C-scheme, please, look on the following simple facts:

1. **a bare physical (massless!) quantity depends on the bare coupling constant, say, α_s^B;**

2. **its renormalization is done with the replacement $\alpha_s^B = Z_a \alpha_s$;**

3. **the charge renormalization constant Z_a depends on the five-loop coefficient in the β-function—β_5—starting from the *fifth* order, α_s^5;**

4. **as a result the renormalized physical quantity starts to "feel" β_5 only at astonishingly large *sixth* order in α_s;**

5. **for the case of the scalar correlator the contribution of order α_s^6 corresponds to the fabulously large 7-loop level**

Explanation of the mystery: the ζ_4 term in the β_5 is, in fact, not independent and not genuinely 5-loop but meets a simple factorization formula ($F^{\zeta_i} = \lim_{\zeta_i \to 0} \frac{\partial}{\partial \zeta_i} F$):

$$\beta_5^{\zeta_4} = \frac{9}{8} \beta_1 \, \beta_4^{\zeta_3}$$

The factorization is not trivial at all:

$$\beta_1 \qquad\qquad\qquad\qquad (\partial/\partial\zeta_3)\beta_4$$

$$\begin{aligned}
\frac{\partial}{\partial \zeta_4}\beta_5 \quad = \quad & \frac{9}{8}\Big(\frac{4}{3}\, n_f\, T_F - \frac{11}{3}\, C_A\Big)\Big(\frac{44}{9}\, C_A^4 - \frac{136}{3}\, C_A^3\, n_f\, T_F \\
& + \frac{656}{9}\, C_A^2\, C_F\, n_f\, T_F - \frac{224}{9}\, C_A^2\, n_f^2\, T_F^2 - \frac{352}{9}\, C_A\, C_F^2\, n_f\, T_F \\
& - \frac{448}{9}\, C_A\, C_F\, n_f^2\, T_F^2 + \frac{704}{9}\, C_F^2\, n_f^2\, T_F^2 - \frac{704}{3}\, \frac{d_A^{abcd}\, d_A^{abcd}}{N_A} \\
& + \frac{1664}{3}\, \frac{d_F^{abcd}\, d_A^{abcd}}{N_A}\, n_f - \frac{512}{3}\, \frac{d_F^{abcd}\, d_F^{abcd}}{N_A}\, n_f^2\Big)
\end{aligned}$$

π-structure of the master p-integrals

We will call a (bare) L-loop p-integral $F(Q^2, \epsilon)$ π-*safe* if the π-dependence of its pole in ϵ *and* constant part can be completely absorbed into the properly defined "hatted" odd zetas.

The first observation of a non-trivial class of π-safe p-integrals — all 3-loop ones — was made in /Broadhurst (1999)/ An extension of the observation on the class of all 4-loop p-integrals was performed in /Baikov, K.Ch. (2010)/ Here it was shown that, given an arbitrary 4-loop p-integral, its pole in ϵ *and* constant part depend on even zetas *only* via the following combinations:

$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6, \;\; \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2}\zeta_6 \quad \text{and} \quad \hat{\zeta}_7 := \zeta_7.$$

Exact meaning for a 4-loop p-integral F_4:

$$F_4(\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7) = F_4(\hat{\zeta}_3, 0, \hat{\zeta}_5, 0, \hat{\zeta}_7) + \mathcal{O}(\epsilon) \qquad \star$$

A generalization of the $\star$ for L=5 has been recently constructed in /Georgoudis, Goncalves, Panzer, Pereira, [1802.00803]/

$\hat{G}$-scheme

Let us define the $\widehat{\text{G}}$-scheme by pretending that hatted zetas do not depend on ϵ. This means that all p-integrals are assumed to be expressed in term of the hatted zetas and that the extraction of the pole part of a p-integral is defined as:

$$\hat{K}\Big(\mathcal{P}(\epsilon)\prod_j \hat{\zeta}_j\Big) := \left(\sum_{i<0} \mathcal{P}_i\,\epsilon^j\right)\prod_j \hat{\zeta}_j,$$

with $\mathcal{P}(\epsilon) = \sum_i \epsilon^i \mathcal{P}_i$ being a polynomial in ϵ with rational coefficients. The corresponding coupling constant will be denoted as $\hat{a}$.

The $\hat{G}$-scheme has some remarkable features. Indeed, one can see just from its definition that the corresponding "hatted" Green function, ADs and Z-factors can be obtained from the normal (that is computed with the G-scheme) by very simple rules.

- As a first step we make a formal replacement of the coupling constant a by $\hat{a}$ in every G-renormalized Green function, AD and Z-factor we want to transform to the $\hat{G}$-scheme.
- Renormalized Green function $\hat{F}(\hat{a})$ is obtained from $F(\hat{a})$ by setting to zero *all* even zetas in the latter (both are assumed as taken at $\epsilon = 0$).
- The same rule works for ADs and β-functions.
- If Z is a (G-scheme) renormalization constant then one should not only nullify all even zetas in $Z(\hat{a})$ but also replace every odd zeta term in it with its "hatted" counterpart.

$\hat{G}$-scheme: useful properties and benefits

1. All 2-point (masless, but not necessarily SI) correlators (at least to 5 loops), β-functions and ADs (at least to 6 loops) are π-free in $\hat{G}$-scheme

2. It is more or less obvious that a change of scheme from $\hat{G}$ one to any other π-free(!) scheme will not induce any π-dependence in correlators. Thus, with the help of the $\hat{G}$-scheme the no-π-conjecture is upgraded to a

BIG No-π Theorem

Let F be any L-loop massless correlator and all L-loop p-integrals form a π-safe class. Then F is π-free in any (massless) renormalization scheme for which corresponding β-function and AD γ are both π-free at least at the level of $L+1$ loops.

$\hat{G}$-scheme: constraints on even zetas

Suppose we know a result for an AD $\hat{\gamma} := (\gamma)_{\hat{G}-\text{scheme}}$ as well as the precise way how hatted zetas are related to the normal ones. The infromation should be enough to construct the result in normal, say, $\overline{\text{MS}}$-scheme Thus, all terms proportional to even zetas in γ should be possible to recover. To do this let us consider the relation between $\hat{a}$ and a:

$$\hat{a} = a \left(1 + \sum_{1 \le i \le L} c_i \, a^i \right) ,$$

As the bare charge must not depend on the choice of the renormalization scheme the coefficients c_i are fixed by requiring that

$$Z_a \, a = \hat{Z}_a(\hat{a}) \hat{a}$$

For simplicity we start from the case of 4 loops. On general grounds we can write

$$\beta = \beta_1 a + \beta_2 a^2 + (r_3 + \beta_3^{\zeta_3} \, \zeta_3) \, a^3 + (r_4 + \beta_4^{\zeta_3} \, \zeta_3 + \beta_4^{\zeta_4} \, \zeta_4 + \beta_4^{\zeta_5} \, \zeta_5) \, a^4$$

where r_i is β_i with all zetas set to zero

The corresponding RCs Z_a and $\hat{Z}_a$ read:

$$Z_a = 1 + \frac{a\beta_1}{\epsilon} + a^2\,(\frac{1}{2\epsilon}\,\beta_2 + \frac{1}{\epsilon^2}\,\beta_1^2) + a^3\,(\frac{1}{3\epsilon}\,(r_3 + \beta_3^{\zeta_3}\,\zeta_3) + \frac{7}{6\epsilon^2}\,\beta_1\,\beta_2 + \frac{1}{\epsilon^3}\,\beta_1^3)$$
$$+a^4\,(\frac{1}{4\epsilon}\,(r_4 + \beta_4^{\zeta_3}\,\zeta_3 + \beta_4^{\zeta_4}\,\zeta_4 + \beta_4^{\zeta_5}\,\zeta_5) + \frac{1}{\epsilon^2}\,(\frac{5}{6}\,\beta_1\,r_3 + \frac{5}{6}\,\beta_1\,\beta_3^{\zeta_3}\,\zeta_3 + \frac{3}{8}\,\beta_2^2)$$
$$+\frac{23}{12\epsilon^3}\,\beta_1^2\,\beta_2 + \frac{1}{\epsilon^4}\,\beta_1^4) \tag{1}$$

and

$$\hat{Z}_a = 1 + \frac{\hat{a}}{\epsilon}\,\beta_1 + \hat{a}^2\,(\frac{1}{2\epsilon}\,\beta_2 + \frac{1}{\epsilon^2}\,\beta_1^2) + \hat{a}^3\,(\frac{1}{3\epsilon}\,(r_3 + \beta_3^{\zeta_3}\,\hat{\zeta}_3) + \frac{7}{6\epsilon^2}\,\beta_1\,\beta_2 + \frac{1}{\epsilon^3}\,\beta_1^3)$$
$$+\hat{a}^4\,(\frac{1}{4\epsilon}\,(r_4 + \beta_4^{\zeta_3}\,\hat{\zeta}_3 + \beta_4^{\zeta_5}\,\hat{\zeta}_5) + \frac{1}{\epsilon^2}\,(\frac{5}{6}\,\beta_1\,r_3 + \frac{5}{6}\,\beta_1\,\beta_3^{\zeta_3}\,\hat{\zeta}_3 + \frac{3}{8}\,\beta_2^2)$$
$$+\frac{23}{12\epsilon^3}\,\beta_1^2\,\beta_2 + \frac{1}{\epsilon^4}\,\beta_1^4). \tag{2}$$

Equation for c_i can be now easily solved with the result

$$
\begin{aligned}
c_1 &= c_2 = 0, \\
c_3 &= -\frac{1}{2}\beta_3^{\zeta_3}\zeta_4 + \frac{5\epsilon^2}{6}\beta_3^{\zeta_3}\zeta_6 - \frac{7\epsilon^4}{2}\beta_3^{\zeta_3}\zeta_8, \\
c_4 &= \frac{1}{4\epsilon}(\beta_4^{\zeta_4} - \beta_1\beta_3^{\zeta_3})\zeta_4 - \frac{3}{8}\beta_4^{\zeta_3}\zeta_4 - \frac{5}{8}\beta_4^{\zeta_5}\zeta_6 \\
&+\frac{5\epsilon}{12}\beta_1\beta_3^{\zeta_3}\zeta_6 + \epsilon^2(\frac{5}{8}\beta_4^{\zeta_3}\zeta_6 + \frac{35}{16}\beta_4^{\zeta_5}\zeta_8) - \frac{7\epsilon^3}{4}\beta_1\beta_3^{\zeta_3}\zeta_8 - \frac{21\epsilon^4}{8}\beta_4^{\zeta_3}\zeta_8
\end{aligned}
$$

As the coefficients c_i have to be finite at $\epsilon \to 0$ we arrive at the exact connection

$$\boldsymbol{\beta_4^{\zeta_4} = \beta_1\,\beta_3^{\zeta_3}}$$

Repeating the same reasoning for L=5 and 6 (and similar one for the case of an AD) we arrive at a host of new exact identities for even zetas terms

Model independent predictions for β and γ for any 1-charge theory

$$\beta_4^{\zeta_4} = \beta_1 \beta_3^{\zeta_3}$$

$$\beta_5^{\zeta_4} = \frac{1}{2}\beta_3^{\zeta_3}\beta_2 + \frac{9}{8}\beta_1\beta_4^{\zeta_3}$$

$$\beta_5^{\zeta_6} = \frac{15}{8}\beta_1\beta_4^{\zeta_5}$$

$$\beta_5^{\zeta_3\zeta_4} = 0$$

$$\beta_6^{\zeta_4} = \frac{3}{4}\beta_2\beta_4^{\zeta_3} + \frac{6}{5}\beta_1\beta_5^{\zeta_3}$$

$$\beta_6^{\zeta_6} = \frac{5}{4}\beta_2\beta_4^{\zeta_5} + 2\beta_1\beta_5^{\zeta_5} - \beta_1^3\beta_3^{\zeta_3}$$

$$\beta_6^{\zeta_3\zeta_4} = \frac{12}{5}\beta_1\beta_5^{\zeta_3^2}$$

$$\gamma_4^{\zeta_4} = -\frac{1}{2}\beta_3^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_1\gamma_3^{\zeta_3}$$

$$\gamma_5^{\zeta_4} = -\frac{3}{8}\beta_4^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_2\gamma_3^{\zeta_3} - \beta_3^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_1\gamma_4^{\zeta_3}$$

$$\gamma_5^{\zeta_6} = -\frac{5}{8}\beta_4^{\zeta_5}\gamma_1 + \frac{5}{2}\beta_1\gamma_4^{\zeta_5}$$

$$\gamma_5^{\zeta_3\zeta_4} = 0$$

$$\gamma_6^{\zeta_4} = \frac{3}{2}\beta_3^{(1)}\gamma_3^{\zeta_3} - \frac{3}{10}\beta_5^{\zeta_3}\gamma_1 - \frac{3}{4}\beta_4^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_2\gamma_4^{\zeta_3} - \frac{3}{2}\beta_3^{\zeta_3}\gamma_3^{(1)} + \frac{3}{2}\beta_1\gamma_5^{\zeta_3}$$

$$\gamma_6^{\zeta_6} = -\frac{1}{2}\beta_5^{\zeta_5}\gamma_1 - \frac{5}{4}\beta_4^{\zeta_5}\gamma_2 + \frac{5}{2}\beta_2\gamma_4^{\zeta_5} + \frac{5}{2}\beta_1\gamma_5^{\zeta_5} + \frac{3}{2}\beta_1^2\beta_3^{\zeta_3}\gamma_1 - \frac{5}{2}\beta_1^3\gamma_3^{\zeta_3}$$

$$\gamma_6^{\zeta_3\zeta_4} = -\frac{3}{5}\beta_5^{\zeta_3^2}\gamma_1 + 3\beta_1\gamma_5^{\zeta_3^2}$$

$$\beta_6^{\zeta_8} = \frac{14}{5}\beta_1\beta_5^{\zeta_7}$$

$$\gamma_6^{\zeta_8} = -\frac{7}{10}\beta_5^{\zeta_7}\gamma_1 + \frac{7}{2}\beta_1\gamma_5^{\zeta_7}$$

$$\beta_6^{\zeta_3\zeta_6} = 0$$

$$\gamma_6^{\zeta_3\zeta_6} = 0$$

$$\beta_6^{\zeta_4\zeta_5} = 0$$

$$\gamma_6^{\zeta_4\zeta_5} = 0$$

The above constraints have been sucessfully checked on the following examples:

L=4 and 5: numerous QCD RG functions (including gauge-dependent ones taken in the Landau gauge) recently computed in

/K.Ch, Falcioni, Herzog and J Vermaseren [1709.08541] .

L=4,5 and 6: β-function and ADs of $O(n)$ ϕ^4 model recently computed in
Batkovich, K. Ch. and Kompaniets, [1601.01960] (γ_2 only)
Schnetz, [1606.08598] ($\beta, \gamma_2, \gamma_m$)
Kompaniets and Panzer, [1705.06483] ($\beta, \gamma_2, \gamma_m$)

Predictions for 6-loop QCD RG functions:

$$\beta_6 \stackrel{\pi}{=} \boxed{\frac{608}{405}\, n_f^5\, \zeta_4} + n_f^4\,(\frac{164792}{1215}\,\zeta_4 - \frac{1840}{27}\,\zeta_6) + n_f^3\,(-\frac{4173428}{405}\,\zeta_4 + \frac{1800280}{243}\,\zeta_6)$$
$$+n_f^2\,(\frac{68750632}{405}\,\zeta_4 - \frac{13834700}{81}\,\zeta_6) + n_f\,(-\frac{146487538}{135}\,\zeta_4 + \frac{40269130}{27}\,\zeta_6)$$
$$+99\,(44213\,\zeta_4 - 64020\,\zeta_6)$$

$$\gamma_6^m \stackrel{\pi}{=} \boxed{\frac{320}{243}\, n_f^5\, \zeta_4 + n_f^4\,(-\frac{90368}{405}\,\zeta_4 + \frac{22400}{81}\,\zeta_6)}$$
$$+n_f^3\,(-\frac{92800}{27}\,\zeta_3\,\zeta_4 - \frac{2872156}{405}\,\zeta_4 + \frac{503360}{243}\,\zeta_6)$$
$$+n_f^2\,(\frac{661760}{9}\,\zeta_3\,\zeta_4 + \frac{155801234}{405}\,\zeta_4 - \frac{378577520}{729}\,\zeta_6 + \frac{12740000}{81}\,\zeta_8)$$
$$+n_f\,(-\frac{1413280}{3}\,\zeta_3\,\zeta_4 - \frac{4187656168}{1215}\,\zeta_4 + \frac{5912758120}{729}\,\zeta_6 - \frac{96071360}{27}\,\zeta_8)$$
$$+3194400\,\zeta_3\,\zeta_4 + \frac{272688530}{81}\,\zeta_4 - \frac{6778602160}{243}\,\zeta_6 + 15889720\,\zeta_8$$

boxed terms are in FULL AGREEMENT with (about 20 years old) results by /Gracey (1996)/ and /Ciuchini, Derkachov, Gracey and Manashov (1999-2000)/

all other terms are new

Conclusions

- We have demonstrated that all π-dependent terms in a generic (L+1)-loop $\overline{\text{MS}}-$ (or, equivalently, G-) anomalous dimension γ are completely fixed by π-independent contributions to γ (and corresponding β) with loop number L or less *provided* the (all) L-loop p-master integrals are π-safe
- The π-safeness holds for L=4 and L=5 and, probably, for L=6. It is known that for L=7 the property (partially) stops to be valid[★] and, thus, our predictions should be modified (at astronomically large for QCD level of **L=8** RG functions)
- All available results at 5 (QCD), and 6 loops (large n_f QCD and the ϕ^4-model) do meet all the constraints we have obtained
- The no-π conjecture for all one-scale RG-invariant Euclidean correlators first suggested Jamin and Miravitllas less than a year ago has been proved and extended to a case of generic Euclidean correlators

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[★] communicated to us by Oliver Schnetz

(the problem is an appearence of the ζ_{12} as indepenent term of some 7-loop finite p-integral, see works by (F.Brown, O.Schnetz, E.Panzer ... on Feynman periods)