Gravity, Scale Invariance and the Hierarchy Problem

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Overview

Drowning by numbers

The fact is that

$\dfrac{G_F \hbar^2}{G_N c^2} \sim 10^{33}$, where G_F — Fermi constant, G_N — Newton constant

Two aspects of the hierarchy problem (**G. F. Giudice'08**):

- "classical"
- "quantum":

  Let M_X be some heavy mass scale. Then one expects

  $$\delta m_{H,X}^2 \sim M_X^2 .$$

  Even if one assumes that there are no heavy thresholds beyond the EW scale, then, naively,

  $$\delta m_{H,grav.}^2 \sim M_P^2 .$$

EFT approach and beyond

A common approach to the hierarchy problem lies within the effective field theory framework:

Low energy description of Nature, provided by the SM, can be affected by an unknown UV physics only though a finite set of parameters

This "naturalness principle" is questioned now in light of the absence of signatures of new physics at the TeV scale. (**G. F. Giudice'13**)

What if one goes beyond the EFT approach? Many examples of non-perturbative phenomena are suggested:

- Multiple Point Criticality Principle (**D. L. Bennett, H. B. Nielsen'94; C. D. Froggatt, H. B. Nielsen'96**):
- Asymptotic safety of gravity (**S. Weinberg'09; M. Shaposhnikov, C. Wetterich'09**)
- EW vacuum decay (**V. Branchina, E. Messina, M. Sher'14; F. Bezrukov, M. Shaposhnikov'14**)

One can attempt to resolve the hierarchy problem (the "classical" part of it) by looking for some non-perturbative effect relating the EW and the Planck-scale physics.

Framework

The absence of an explicit UV completed theory encompassing the SM and GR makes our analysis ambiguous. To narrow the window of possibilities, one adopts the conjectures:

- Scale Invariance
    - The idea of reducing an amount of dimensionfull parameters as a way towards the fundamental theory seems fruitful (**J. Garcia-Bellido, J. Rubio, M. Shaposhnikov, D. Zenhausern'11**)
- No degrees of freedom with mass scales above the EW scale
    - Experimental data?
- Dynamical gravity
    - We believe gravity plays a crucial role in the effect we look for

We are interested in simple models comprising the scalar fields and gravity, on which one can test the non-perturbative mechanism. We do not argue that these models can indeed be embedded into the complete theory. However, the successful mechanism can be viewed as an argument in favour of those properties of the theory which support its existence.

Outline of the idea

Consider a theory containing the scalar field φ of a unit mass dimension, the metric field $g_{\mu\nu}$ and, possibly, other fields denoted collectively by $\mathcal{A}$.

Time-independent, spatially homogeneous vev of φ

$\langle\varphi\rangle = Z^{-1} \int \mathcal{D}\varphi \mathcal{D}g_{\mu\nu} \mathcal{D}\mathcal{A} \, \varphi(0) e^{-S}$, where $Z = \int \mathcal{D}\varphi \mathcal{D}g_{\mu\nu} \mathcal{D}\mathcal{A} \, e^{-S}$,

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and S is the euclidean action of the theory.

Let us reorganize the numerator in the expression for $\langle\varphi\rangle$ by making

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Change of the field variable

$\varphi \to \varphi_0 e^{\bar{\varphi}}$ at $\varphi \gtrsim \varphi_0$,

---

where φ_0 is an appropriate scale of the theory. Then,

$\int_{\varphi \gtrsim \varphi_0} \mathcal{D}\varphi \, \varphi(0) e^{-S} \to \varphi_0 \int_{\bar{\varphi} \gtrsim 0} \mathcal{D}\bar{\varphi} J e^{-W}$,

where $W = -\bar{\varphi}(0) + S$ and J is a Jacobian of the transformation.

Outline of the idea

Suppose that the functional W admits appropriate saddle points through which the modified path integral can be evaluated. Then, in the leading order saddle-point approximation (SPA),

> **Vev of φ computed via W**
>
> $\langle \varphi \rangle \sim \varphi_0 e^{-\bar{W} + S_0},$

where $\bar{W}$ is the value of W computed at a saddle and S_0 is the value of S computed at the ground state.

For this to work, it is necessary to find

- Appropriate saddle points of the functional W,
- Semiclassical parameter that would justify the SPA,
- Physical argumentation that would justify the change of the field variable.

In case if the vacuum geometry is not flat, boundary terms must also be included into consideration.

The quantity $\bar{W} - S_0$ can be viewed as a rate of suppression of the classical scale φ_0. If this rate is large and positive, the hierarchy of scales emerges.

The Dilaton model

Later we will make use of a special type of instanton configurations, which appear in (asymptotically) SI theories with non-minimal couplings of scalar fields to gravity. Let us first discuss the properties of these instantons using a simple toy model.

Ingredients of the model: Gravity, one scalar dof non-minimally coupled to gravity.

(Euclidean) Lagrangian

$$\frac{\mathcal{L}}{\sqrt{g}} = -\frac{1}{2}\xi\varphi^2 R + \frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4}\varphi^4 \text{ with } \xi > 0$$

Boundary term

$$I = -\int d^3x\sqrt{\gamma}K\xi\varphi^2$$

The classical ground state of the model, $\varphi = \varphi_0$, $\quad R = \dfrac{\lambda\varphi_0^2}{\xi}$, breaks SI spontaneously.

The Dilaton model

To analyze classical configurations, it is convenient to rewrite the model in the form in which the non-minimal coupling is absent. To this end, one performs the Weyl transformation,

$$\varphi = \varphi_0 \Omega \,, \quad \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \,, \quad \Omega = e^{\frac{\bar{\varphi}}{\sqrt{\xi}\varphi_0}} \,.$$

Then,

Lagrangian in the Einstein frame

$$\frac{\tilde{\mathcal{L}}}{\sqrt{\tilde{g}}} = -\frac{1}{2}\xi\varphi_0^2 \tilde{R} + \frac{1}{2a}(\tilde{\partial}\bar{\varphi})^2 + \frac{\lambda}{4}\varphi_0^4 \,, \quad a = \frac{1}{6 + 1/\xi}$$

Boundary term in the Einstein frame

$$I_{GH} = -\xi\varphi_0^2 \int d^3x \sqrt{\tilde{\gamma}}\tilde{K}$$

We identify the SI breaking scale with the Planck mass: $M_P \equiv \sqrt{\xi}\varphi_0$.

Classical configurations in the Dilaton model

We consider the $O(4)$-symmetric case.

Metric ansatz

$d\bar{s}^2 = f(r)^2 dr^2 + r^2 d\Omega_3^2$

Equations of motion

$$\partial_r\left(\frac{r^3\bar{\varphi}'}{af}\right) = 0 \,, \quad \frac{1}{f^2} = 1 + \frac{a}{6M_P^2}\frac{r^2}{f^2}\bar{\varphi}'^2 \pm b^2 r^2 \,, \quad b^2 = \frac{|\lambda|M_P^2}{12\xi^2}$$

---

where the plus (minus) sign in the second expression holds for negative (positive) λ.

Thanks to the form of the metric ansatz, the 00-component of the Einstein equations reduces to an algebraic equation for f.

An obvious solution of EoM: $\bar{\varphi} = 0 \,, \quad f^2 = \dfrac{1}{1 \pm b^2 r^2}$ — the ground state.

Classical configurations in the Dilaton model

Consider now configurations obeying the equation $\dfrac{r^3 \bar{\varphi}'}{af} = C$ with C some non-zero constant. Let they approach the ground state at large distances.

The short-distance asymptotics of such configurations is

$$\bar{\varphi} \sim -\gamma M_P \log(M_P r) \,, \quad \tilde{R} \sim a M_P^{-4} r^{-6} \,, \quad \gamma = \sqrt{6a} \,, \quad r \to 0.$$

One observes the physical singularity at $r = 0$. Hence, we need the source of the field $\bar{\varphi}$ at that point:

$$S \quad \to \quad W = S - \int d^4x\, j(x) \bar{\varphi}(x).$$

Let $j(x) = M_P^{-1} \delta^{(4)}(x)$. Then $C = -M_P^{-1}$, and the configuration obeys

$$\bar{\varphi}' = -\frac{af}{r^3 M_P} \,, \quad \frac{1}{f^2} = 1 + \frac{a}{6 M_P^4 r^4} \pm b^2 r^2$$

We will refer to such configurations as "singular instantons".

The configurations of this type were studied before in the context of the cosmological initial value problem, (**S. W. Hawking, N. Turok'98**)
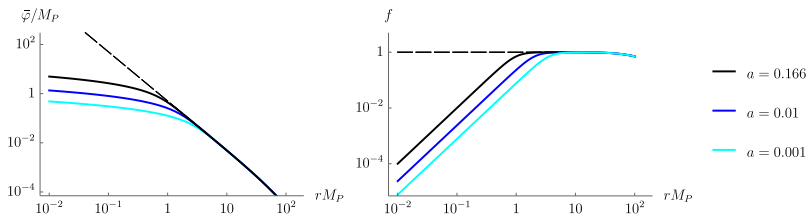
Singular instanton in the Dilaton model

Figure: The profile of the singular instanton at different values of a. We choose the Anti-de Sitter asymptotic geometry with $b = 0.01$.

We see that gravity softens the divergence of the scalar field as compared to the flat space limit.

The characteristic size of the instanton (the size of its "core") is $r_* = a^{1/4} M_P^{-1}$. We will assume the good separation between r_* and the "cosmological" length b^{-1}. It holds provided that

$$bM_P^{-1} \ll 1.$$

Instanton action in the Dilaton model

Let us evaluate the euclidean action $\bar{S}$ and the boundary term $\bar{I}_{GH}$ of the instanton, relative to the euclidean action S_0 and the boundary term $I_{GH,0}$ of the ground state.

Net action

$\bar{S} - S_0 \sim ab^2 M_P^{-2} \ll 1$

Net boundary term

$$\bar{I}_{GH} - I_{GH,0} \sim \left\{ \begin{array}{ll} 0 \, , & \lambda > 0 \\ a^{-1} M_P^{-2} r_s^{-2} \to 0 \, , & r_s \to \infty \, , \quad \lambda = 0 \\ a^{-1} b^{-1} M_P^{-2} r_s^{-3} \to 0 \, , & r_s \to \infty \, , \quad \lambda < 0 \end{array} \right.$$

We conclude that the nontrivial background geometry does not lead to a significant contribution to the net instanton action. Hence, in proceeding with the classical analysis in more complicated models, we can focus solely on the core region of the instanton.

One Example: Higgs+Gravity

Let us study the case when gravity breaks explicitly the SI of the theory, and we demand the Planck mass to be the only classical dimensional parameter in the theory.

For example, take the following Lagrangian describing the dynamics of the Higgs and the metric fields

General Lagrangian

$$\frac{\mathcal{L}_{\phi,g}}{\sqrt{g}} = G_4(|\phi|)R + G_2(|\phi|,|\partial\phi|^2)$$

where the functions G_4, G_2 are chosen so that to reproduce the SM Higgs kinetic term, the Higgs field potential with $m_H = 0$, and GR in the low-energy limit.

After all, this must be supplemented with the rest of the SM content. The latter is not important for the analysis of classical configurations made of the Higgs and the metric fields.

The "probe" Lagrangian

To start with, take the model resembling the Dilaton theory:

Jordan frame Lagrangian

$$\frac{\mathcal{L}_{\varphi,g}}{\sqrt{g}} = -\frac{1}{2}(M_P^2 + \xi\varphi^2)R + \frac{1}{2}(\partial\varphi)^2 + V(\varphi), \text{ where } V(\varphi) = \frac{\lambda}{4}\varphi^4 \text{ and } \xi > 0$$

The Weyl rescaling $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $\Omega^2 = M_P^{-2}(M_P^2 + \xi\varphi^2)$ gives

Einstein frame Lagrangian

$$\frac{\mathcal{L}_{\varphi,g}}{\sqrt{\tilde{g}}} = -\frac{1}{2}M_P^2\tilde{R} + \frac{1}{2a(\varphi)}(\tilde{\partial}\varphi)^2 + \tilde{V}(\varphi),$$

where $a(\varphi) = \dfrac{\Omega^4}{\Omega^2 + 6\xi^2\varphi^2/M_P^2}$ and $\tilde{V}(\varphi) = V(\varphi)\Omega^{-4}$.

Mechanism: step 1

According to the discussion above, the first step in the course of evaluating the vev $\langle \varphi \rangle$ is to

- Change the field variable:

$$\varphi \to M_P e^{\bar{\varphi}/M_P}$$

  in the SI regime $\varphi \gg M_P/\sqrt{\xi}$. **Motivation**: it is $\bar{\varphi}$ which carries the valid dof in this regime.

  To find $\langle \varphi \rangle$, one should search for saddles of the functional $W = -\bar{\varphi}(0) + S$, where S is the euclidean action of the theory. They are singular instantons of the kind studied above.

Equation of motion for the variable $\bar{\varphi}$ in the SI regime

$$\frac{r^3 \bar{\varphi}'}{a_{SI} f} = -\frac{1}{M_P} \ , \quad a_{SI} = a(\varphi \gg M_P/\sqrt{\xi}) = \frac{1}{6 + 1/\xi}$$

Asymptotics of the instanton at short distances

$$\bar{\varphi} \sim -M_P \sqrt{6a_{SI}} \log r M_P \ , \quad r \to 0$$

Mechanism: problems

In trying to compute the instanton value of W, one encounters difficulties:

- $\bar{\varphi}(0) = \infty$. How to treat this divergence?
- Where is the semiclassical parameter?
- The instanton action turns out to be small (in agreement with the discussion above). How to make it large?

To overcome these issues, we switch on a high-energy large-field content of the model. **Motivation**: one can think of the "probe" Lagrangian as describing an effective theory with the cutoff $\Lambda \sim M_P/\xi$. This justifies the usage of the Planck-suppressed in the low-$\bar{\varphi}$ limit operators which are composed of the scalar and metric fields.
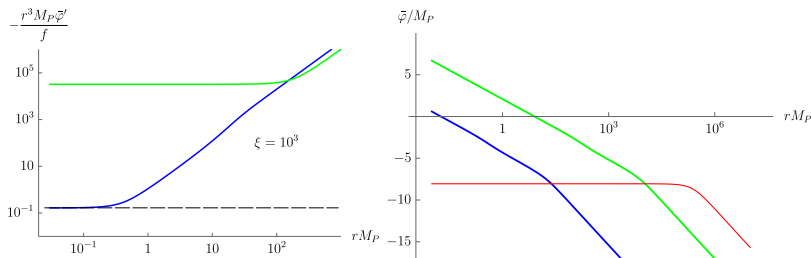
Mechanism: problems

Figure: The blue line represents the configuration obeying the boundary condition imposed by the source. Hence, it is a valid singular instanton. The configuration painted green is the one with the large euclidean action; for illustration, we choose for it $\bar{S}_E = 40$. The potential is taken as in the SM with the central values of the parameters. For illustrative purposes, the bounce is also plotted in red.

Mechanism: step 2

- To shape the behaviour of the instanton in the large-field limit, one makes use of the operators respecting the (asymptotic) shift symmetry of the theory.

  For example, consider the effect of the following

Derivative operator

$\mathcal{O}_n = \sqrt{g}\,\delta_n \dfrac{(\partial\varphi)^{2n}}{(M_P\Omega)^{4n-4}}$ and let $n = 2$ for simplicity

Then,

Modified equation of motion for the variable $\bar{\varphi}$ in the SI regime

$$\frac{4\delta}{M_P^4}\frac{r^3\bar{\varphi}'^3}{f^3} + \frac{1}{a_{SI}}\frac{r^3\bar{\varphi}'}{f} = -\frac{1}{M_P}$$

Mechanism: step 2

Let $\bar{r}$ be the size of the region where the quartic derivative term dominates,

$$\bar{r} \sim M_P^{-1} \delta^{1/6} a_{SI}^{1/2} \; .$$

We assume that $\bar{r}$ is smaller than the characteristic length at which $a(\varphi)$ varies.

At $r \lesssim \bar{r}$, the asymptotics of the instanton changes as

Modified asymptotics of the instanton at short distances

$$\bar{\varphi}' \sim M_P^2 \delta^{-1/6} \; , \quad f \sim r M_P \delta^{1/6} \; , \quad r \lesssim \bar{r}$$

Thus, *the scalar field is not divergent any more.*[1] Its magnitude at the center of the instanton is

$$\bar{\varphi}(0)/M_P \sim a_{SI}^{1/2}(\log \delta - 3 \log a_{SI} + \mathcal{O}(1)) \; .$$

The first issue is, therefore, cured. The other two remain.

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[1] This holds regardless the particular derivative structure modifying the "probe" Lagrangian in the high-energy limit.

Mechanism: step 3

- The strength a_{SI} of the scalar field source must be enhanced in the region probed by the core of the singular instanton:

$$a_{SI} \to a_{HE} = \frac{1}{\varkappa/\xi + 6} , \quad r \to 0 , \quad \bar{\varphi} \gtrsim M_P , \quad \varkappa > -1/\xi$$

This implies the blow-up of the curvature of the field space.

As an example, consider the following

Lagrangian

$$\frac{\mathcal{L}_{\varphi, g}}{\sqrt{g}} = -\frac{M_P^2}{2} F(\varphi/M_P) R + \frac{1}{2} G(\varphi/M_P)(\partial\varphi)^2 + \delta\xi^2 \frac{(\partial\varphi)^4}{(M_P\Omega)^4} + \frac{\lambda}{4} \varphi^4$$

$$F = 1 + \xi\varphi^2/M_P^2 , \quad G = \frac{1 + \varkappa\varphi^2/M_P^2}{1 + \varphi^2/M_P^2} \text{ and } \kappa \text{ a constant}$$

Then, the coefficient a_{SI} becomes field-dependent,

Field-dependent source strength

$$a_{SI} = \frac{1}{\alpha/\xi + 6} \text{ where } \alpha = \frac{1}{2}(1 - \tanh(\bar{\varphi}/M_P)) + \frac{\varkappa}{2}(1 + \tanh(\bar{\varphi}/M_P))$$

Mechanism: step 3

If $a_{HE} \gg 1$, then

$$\bar{W} \sim \sqrt{a_{HE}} \, , \qquad \langle \varphi \rangle \sim M_P e^{-\bar{W}}$$
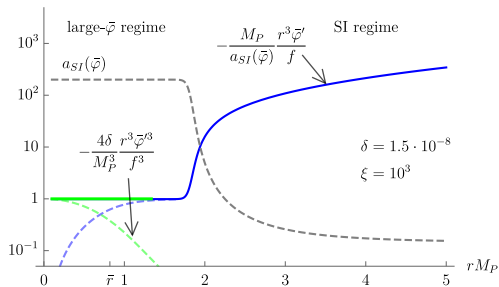

Figure: The relevant combinations of fields in the core region of the instanton. One observes that by enhancing the source (dashed line), one can make equation of motion satisfied in the large-$\bar{\varphi}$ limit.

Implications for the Hierarchy problem

- The real scalar field φ is identified with the Higgs field degree of freedom in the unitary gauge,

$$\phi = 1/\sqrt{2}\, (0, \varphi)^T \,.$$

- The Higgs-gravity Lagrangian is supplemented with the rest of the low-energy content of the theory. Fluctuations of the fields affect the prefactor in the leading-order formula

$$v = M_P e^{-\bar{W}} \,.$$

The validity of the SPA enables us to believe that the higher-order corrections do not change drastically the leading-order calculation.

- One should modify the Higgs coupling to the gauge fields in order to prevent them from becoming tachyonic at large fields.

Further examples: Higgs+Dilaton+Gravity

- One can as well study the SI theory of two scalar fields coupled to gravity in a non-minimal way.

- The Planck scale appears as a result of a spontaneous breaking of the SI by one of the fields.

- By studying singular instantons similar to those appeared before, one can evaluate their contribution to the vev of the second field.

- In choosing a particular model for investigation, one demands it to be convertible into a phenomenologically viable theory upon identifying one of its scalar fields with the Higgs field dof and supplementing it with the rest of the SM content.

Discussion

- The mechanism allows to generate an exponentially small ratio of scales without a fine-tuning among the parameters of the theory, but it does not explain a particular value of this ratio.

- The mechanism works in a broad class of scalar-tensor theories.

- It seems that in many cases the properties of the theory at low energies are irrelevant for the mechanism, since it operates essentially in the Planck region. But it is possible to construct a counterexample.

- Approximate asymptotic Weyl invariance is needed for the successful implementation of the mechanism.

Outlook

- Meaning of W?

- Self-consistency

- Fluctuations above the instanton

- Correlation functions in the scalar sector via many-instanton configurations

Thank you!

Model setup

We are looking for the model which

- is convertible into a phenomenologically viable theory upon identifying one of its scalar fields with the Higgs field dof and supplementing it with the rest of the SM content,
- enjoys global SI,
- allows for desirable instantons studied earlier.

This motivates us to introduce the following

Lagrangian

$$
\frac{\mathcal{L}}{\sqrt{g}} = -\frac{1}{2}\mathcal{G}(\vec{\varphi})R + \frac{1}{2}\gamma_{ij}^{(2)}(\vec{\varphi})g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j
$$
$$
+ \sum_{n=2}^{\infty}\gamma_{i_1,\ldots,i_{2n}}^{(2n)}(\vec{\varphi})g^{\mu\nu}\partial_\mu\varphi^{i_1}\partial_\nu\varphi^{i_2}...g^{\rho\sigma}\partial_\rho\varphi^{i_{2n-1}}\partial_\sigma\varphi^{i_{2n}} + V(\vec{\varphi}) ,
$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)^T$.

The classical ground state $\vec{\varphi}_{vac.} = (\varphi_0, 0)^T$.

Model setup

The functions introduced in the Lagrangian are taken as follows,

$$
\begin{aligned}
\mathcal{G} &= \xi_1 \varphi_1^2 + \xi_2 \varphi_2^2 \,, \\
\gamma_{ij}^{(2)} &= \delta_{ij} + \varkappa \mathcal{G}\mathcal{F}\mathcal{J}^{-4}(1+6\xi_i)(1+6\xi_j)\varphi_i\varphi_j \,, \\
\gamma_{ijkl}^{(4)} &= \delta \mathcal{J}^{-8}(1+6\xi_i)(1+6\xi_j)(1+6\xi_k)(1+6\xi_l)\varphi_i\varphi_j\varphi_k\varphi_l \,, \\
\gamma_{i_1 \ldots i_{2n}}^{(2n)} &= 0 \,, \quad n > 2 \,.
\end{aligned}
$$

Here

$$
\begin{aligned}
\mathcal{J}^2 &= (1+6\xi_1)\varphi_1^2 + (1+6\xi_2)\varphi_2^2 \,, \\
\mathcal{F} &= \frac{(1+6\xi_1)\varphi_2^2}{(1+6\xi_2)\varphi_1^2 + (1+6\xi_1)\varphi_2^2} \,,
\end{aligned}
$$

and we take $\xi_2 > \xi_1 > 0$, $\delta > 0$. The potential for the scalar fields is chosen as

$$
V = \frac{\lambda}{4}\varphi_2^4 \,.
$$

On the classical ground state $\mathcal{G}(\vec{\varphi}_{vac.}) = \xi_1 \varphi_0^2 \equiv M_P^2$.

Polar field variables ($\varkappa = 0$)

The change of variables $\rho = \dfrac{M_P}{2} \log \dfrac{\mathcal{J}^2}{M_P^2}$, $\theta = \arctan\left(\sqrt{\dfrac{1 + 6\xi_1}{1 + 6\xi_2}} \dfrac{\varphi_2}{\varphi_1}\right)$ results in

Lagrangian in terms of the fields ρ and θ

$$\frac{\tilde{\mathcal{L}}}{\sqrt{\tilde{g}}} = -\frac{1}{2} M_P^2 \tilde{R} + \frac{1}{2a(\theta)}(\tilde{\partial}\rho)^2 + \frac{b(\theta)}{2}(\tilde{\partial}\theta)^2 + \delta\frac{(\tilde{\partial}\rho)^4}{M_P^4} + \tilde{V}(\theta)$$

where $a(\theta) = a_0(\sin^2\theta + \zeta\cos^2\theta)$, $\quad b(\theta) = \dfrac{M_P^2\zeta}{\xi_1}\dfrac{\tan^2\theta + \xi_1/\xi_2}{\cos^2\theta(\tan^2\theta + \zeta)^2}$,

$\tilde{V}(\theta) = \dfrac{\lambda M_P^4}{4\xi_2^2}\dfrac{1}{(1 + \zeta\cot^2\theta)^2}$, $\quad \zeta = \dfrac{(1 + 6\xi_2)\xi_1}{(1 + 6\xi_1)\xi_2}$, $\quad a_0 = \dfrac{1}{6 + 1/\xi_2}$.

The inverse formulas

$$\varphi_1 = \frac{M_P\cos\theta}{\sqrt{1 + 6\xi_1}}e^{\rho/M_P}, \qquad \varphi_2 = \frac{M_P\sin\theta}{\sqrt{1 + 6\xi_2}}e^{\rho/M_P}.$$

Singular instanton with $\delta = \varkappa = 0$

The vacuum solution reads as follows,

Classical ground state

$$\rho_{vac.} = \frac{M_P}{2} \log \frac{1 + 6\xi_1}{\xi_1} \ , \quad \theta_{vac.} = 0.$$

Now we look for the saddle points of the functional $W = S - \int d^4x \delta^{(4)}(x)\rho(x)/M_P$.

Then, the equation for the radial field is $\dfrac{\rho' r^3}{f} = -\dfrac{a(\theta)}{M_P}$.

Large-distance asymptotics

$$\rho - \rho_{vac.} \sim r^{-2} \ , \quad \theta \sim r^{-2} \ , \quad r \to \infty.$$

Short-distance asymptotics

$$\rho \sim -\gamma M_P \log M_P r \ , \quad \tilde{R} \sim r^{-6} \ , \quad \frac{\pi}{2} - \theta \sim r^\eta \quad r \to 0,$$

where $\gamma = \sqrt{6a_0}$, $\eta = \sqrt{6a_0(1 - \xi_1/\xi_2)}$, $a_0 = a(\pi/2)$. We will call the configuration obeying these boundary conditions the "singular instanton".
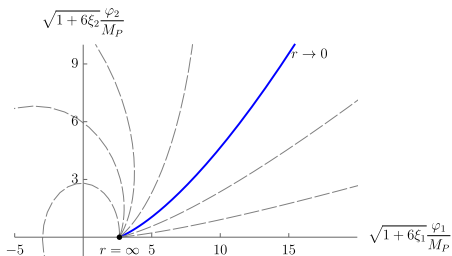
Singular instanton with $\delta = \varkappa = 0$

Figure: The singular instanton in the model with two scalar fields (the solid blue line). The parameters of the model are $\xi_1 = 1$, $\xi_2 = 1.1$ and $\lambda = 0$. Dashed lines are examples of configurations with no definite limit of θ at $r \to 0$.

The asymptotics in terms of the original fields are $\varphi_1 \sim r^{-\gamma+\eta}$, $\varphi_2 \sim r^{-\gamma}$. For $\varkappa = 0$, we have $\eta < \gamma$.

It is important to note that the divergence of φ_1, φ_2 originates fully from the divergence of the radial field ρ.

Regularization of the instanton with $\delta \neq 0$, $\varkappa = 0$

Let us switch on the quartic derivative operator. Then, the equation for the radial field becomes $\dfrac{4\delta}{M_P^4} \dfrac{\rho'^3 r^3}{f^3} + \dfrac{\rho' r^3}{a(\theta)f} = -\dfrac{1}{M_P}$.

We assume that the size $\bar{r}$ of the region where the quartic derivative term dominates is smaller than the characteristic length at which $a(\theta)$ varies. In this case,

$$\bar{r} \sim M_P^{-1} \delta^{1/6} a_0^{1/2} \ .$$

Inside this region,

Modified short-distance asymptotics

$\rho' \sim -M_P^2 \delta^{-1/6} \ , \quad f \sim M_P r \delta^{1/6} \ , \quad \tilde{R} \sim r^{-2}.$

Hence, the radial field is not divergent any more. Its magnitude at the center of the instanton is

$$\rho(0)/M_P \sim a_0^{1/2}(\log \delta - 3 \log a_0 + \mathcal{O}(1)) \ .$$

Regularization of the instanton with $\delta \neq 0$, $\varkappa = 0$
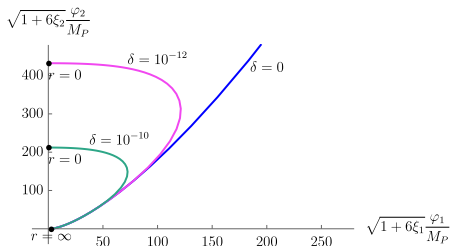

Figure: Regularization of the singular instanton by the higher-dimensional operator. The parameters of the model are $\xi_1 = 1$, $\xi_2 = 1.1$ and $\lambda = 0$.

Note that the small values of δ are required in order to ensure the separation of the region where $a(\theta)$ varies from the region where the regularization acts. This does not bring in the model any new interaction scales below the Planck scale.

Source enhancement with $\varkappa \neq 0$

As before, we try to adjust the parameters of the model so that to permit the large source values. This is achieved if we switch on $\varkappa$. The Lagrangian in the polar field variables remains the same but with $a(\theta)$ replaced by $\tilde{a}(\theta)$ where

$$\frac{1}{\tilde{a}(\theta)} = \frac{1}{a(\theta)} + \varkappa \sin^2 \theta \; .$$

Positive-defineteness requires $\varkappa > \varkappa_{crit.} = -\dfrac{1}{a_0}$.
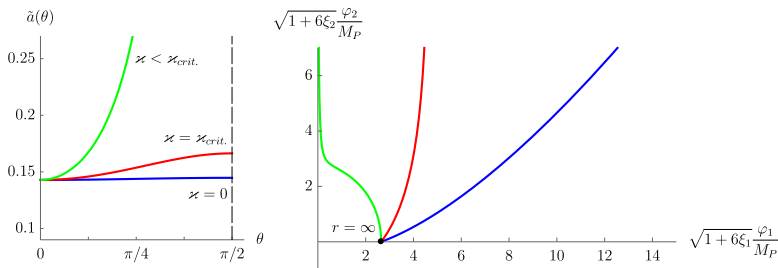

Figure: *Left:* the function $\tilde{a}(\theta)$. The critical value, $\varkappa = \varkappa_{crit.}$, corresponds to $\eta = \gamma$. The value below the critical, $\varkappa < \varkappa_{crit.}$, is chosen so that $\tilde{a}(\theta_0) \equiv \tilde{a}_0 = 100$. This value lies close to the positivity bound. *Right:* the corresponding instanton solutions. At $\varkappa = 0$, the instanton studied before is reproduced. The parameters of the model are $\xi_1 = 1$, $\xi_2 = 1.1$ and $\lambda = 0$.

New scale via the instanton

In the SPA $\langle\varphi_2\rangle \sim M_P e^{-\bar{W}}$. Contributions to $\bar{W}$ come from the source term and the instanton action $\bar{S}$:

$$\bar{W} = -\frac{\rho(0)}{M_P} + \int_0^\infty dr(\bar{\mathcal{L}}_\delta - \bar{\mathcal{L}}_V) \,,$$

where

$$\bar{\mathcal{L}}_\delta = 2\pi^2 r^3 f \left(\frac{\rho'}{M_P f}\right)^4 \,, \quad \bar{\mathcal{L}}_V = 2\pi^2 r^3 f \tilde{V}(\theta) \,.$$

One obtains that
- The potential term, $\mathcal{L}_V$, contributes negligibly (cf. the Higgs-gravity model).
- The total contribution from the short-distance part of the instanton can be of either sign. For example, if we take $\varkappa = 0$, then $\bar{W}$ is positive for $\delta \gtrsim 10^{-10}$. However, in this case it is impossible to achieve $\bar{W} \gg 1$ (see figure).
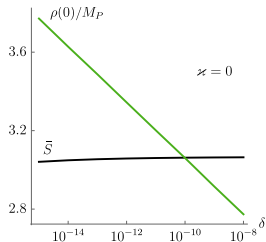

Figure: The singular term $\rho(0)/M_P$ and the instanton action. Here $\varkappa = 0$ and $\xi_1 = 1$, $\xi_2 = 1.1$.

New scale via the instanton

- The desired suppression rate is obtained if one takes large $\tilde{a}_0$ (corresponding to the values of $\varkappa$ close to the positivity bound). In this case
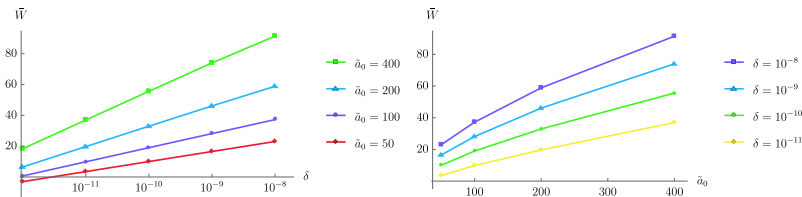
$$\bar{W} \sim \sqrt{\tilde{a}_0} \ . \tag{1}$$

Figure: The suppression rate $\bar{W}$ as a function of δ and $\tilde{a}_0$.

Higgs-Dilaton theory

Higgs-Dilaton Lagrangian

$$\frac{\mathcal{L}_{\chi,\phi}}{\sqrt{g}} = -\frac{1}{2}(\xi_\chi \chi^2 + 2\xi_h \phi^\dagger \phi)R + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}(\partial\phi)^2 + V(\chi,\phi^\dagger\phi)$$

The Potential

$$V(\chi,\phi^\dagger\phi) = \lambda \left(\phi^\dagger\phi - \frac{\alpha}{2\lambda}\chi^2\right)^2 + \beta\chi^4$$

The classical ground state: $h_0^2 = \frac{\alpha}{\lambda}\chi_0^2 + \frac{\xi_h}{\lambda}R$, $\quad R = \frac{4\beta\lambda\chi_0^2}{\lambda\xi_\chi + \alpha\xi_h}$.

The Planck mass $M_P^2 \equiv \xi_\chi\chi_0^2 + \xi_h h_0^2$.

The space of parameters of the theory is subject to phenomenological constraints.
- Non-minimal couplings: $\xi_\chi \ll 1 \ll \xi_h$ (from inflation)
- Couplings in the potential:

$$m_H^2 \sim \frac{\alpha M_P^2}{\xi_\chi} \quad \Rightarrow \quad \alpha \sim 10^{-34}\xi_\chi$$

$$\Lambda \sim \frac{\beta M_P^4}{\xi_\chi^2} \quad \Rightarrow \quad \beta \sim 10^{-56}\alpha^2$$

(**M. Shaposhnikov, D. Zenhausern'08; J. Garcia-Bellido, J. Rubio, M. Shaposhnikov, D. Zenhausern'11**)

Higgs vev generation in the Higgs-Dilaton theory

If one puts $\alpha = 0$ in the Higgs-Dilaton potential, then $m_H = 0$ classically, and the radiative corrections to the Higgs mass do not shift it towards the observed value.

In order for the mechanism to work, one must modify the theory in the limit of large magnitudes and momenta of the Higgs field. This is done by introducing the higher-dimensional operators of the form considered above. They do not spoil phenomenological consequences of the theory.
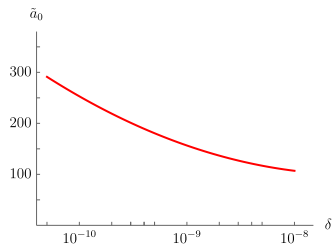

Figure: The set of parameters $(\tilde{a}_0, \delta)$, for which $\bar{W} = \log(M_P/v)$. Here we choose $\xi_\chi = 5 \cdot 10^{-3}$, $\xi_h = 5 \cdot 10^3$ and λ coinciding with the SM running Higgs self-coupling at NNLO with the central values of the top quark and Higgs masses.