Scattering beyond the plane-wave approximation and probing of phases of scattering amplitudes

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Consider a generic scattering problem 2 → N:

\[ \langle p_1 \rangle, \sigma_1 \] \quad [\text{input}] \quad \rightarrow \quad T_{fi} \quad \rightarrow \quad \langle p_2 \rangle, \sigma_2 \] \quad [\text{output}]

The plane-wave approximation is said to be applicable when:

\[ \sigma \ll m \quad \text{the packets are narrow in the momentum space} \]

\[ \sigma \perp \gg \lambda_c = \frac{\hbar}{mc} \quad \text{and wide in the coordinate space} \]
Naively, the non-paraxial corrections to observables are

\[ \sim \frac{\sigma^2}{m^2} \sim \frac{\lambda_c^2}{\sigma_\perp^2} \ll 1 \]

Say, for the LHC beam it is less than \(10^{-22}\)

For modern electron accelerators it is less than \(10^{-14}\) (and some 2-3 orders larger for ILC and CLIC)

For electron microscopes it is less than \(10^{-6}\) (!) Verbeeck, et al. 2011

Fortunately, there are also dynamical (not purely “geometrical”) effects!
The plane-wave approximation in scattering does not work if:

1. The impact parameters are large: an MD-effect (Novisibirsk)  
   \textit{Tikhonov 1982; Kotkin, Serbo, Schiller 1992}

2. The initial particles are unstable  
   \textit{Ginzburg 1996; Melnikov, Serbo, 1997}

3. One describes neutrino oscillations  
   \textit{Akhmedov, Smirnov 2009; Akhmedov, Kopp, 2010}

4. The in-states are not Gaussian (!)  
   \textit{Jentschura, Serbo, 2011; Ivanov, 2011}

5. The quantum coherence is lost (!)  
   \textit{Sarkadi 2016; D.K., Serbo 2017}

To be addressed in this talk
Outline

(1) Some non-Gaussian quantum states:
   I. Vortex photons, electrons and neutrons with orbital angular momentum,
   II. Airy photons and electrons,
   III. Schrödinger cats,
   IV. Their generalizations

(2) Non-paraxial wave packets and the Wigner functions

(3) Non-paraxial effects in scattering:
I. Finite momentum uncertainties and impact-parameter, “approximate” conservation laws, etc.

II. The cross section grows dependent upon a phase of a scattering amplitude
   (say, hadronic or Coulomb one)

III. Quantum decoherence and the Wigner functions' negativity may affect the cross section

IV. Enhancement of the non-paraxial corrections to the plane-wave cross sections
   for vortex particles with large orbital angular momentum
1. Non-Gaussian states: vortex particles with orbital angular momentum (OAM)


Twisted photons: Allen, et al. 1992

A Bessel-state of a free scalar particle:

\[ \psi(r) = N J_\ell(k\rho) e^{-i\varepsilon t + ip_\parallel z + i\ell\phi_r} \]

Probability density for a well-normalized wave packet
1. Non-Gaussian states: vortex particles with orbital angular momentum (OAM)

They form a complete and orthogonal set:

\[
\langle p_\parallel', \kappa', \ell' | p_\parallel, \kappa, \ell \rangle = (2\pi)^2 2\varepsilon(p) \delta(p_\parallel - p_\parallel') \frac{\delta(\kappa - \kappa')}{\kappa} \delta_{\ell\ell'}
\]

\[
\hat{\psi}(x) = \sum_\ell \int \frac{dp_\parallel \kappa d\kappa}{(2\pi)^2 2\varepsilon} (\langle x | p_\parallel, \kappa, \ell \rangle \hat{a}_{\{p_\parallel, \kappa, \ell\}} + \text{H.c.})
\]

\[
= \sum_\ell \int \frac{dp_\parallel \kappa d\kappa}{(2\pi)^2 \sqrt{2\varepsilon}} \left( J_\ell(\kappa \rho) e^{-i\varepsilon t + ip_\parallel z + i\ell \phi_r} \hat{a}_{\{p_\parallel, \kappa, \ell\}} + \text{H.c.} \right)
\]

\[
[\hat{a}_{\{p_\parallel, \kappa, \ell\}}, \hat{a}^\dagger_{\{p_\parallel', \kappa', \ell'\}}] = (2\pi)^2 \delta(p_\parallel - p_\parallel') \frac{\delta(\kappa - \kappa')}{\kappa} \delta_{\ell\ell'}
\]

\[
[\hat{\psi}(x), \hat{\psi}^\dagger(x')] = \sum_\ell \int \frac{dp_\parallel \kappa d\kappa}{(2\pi)^2 2\varepsilon} J_\ell(\kappa \rho) J_\ell(\kappa' \rho') \left( e^{-i\varepsilon(t-t') + ip_\parallel(z-z') + i\ell(\phi_r - \phi_r')} - \text{c.c.} \right).
\]

D.K., PRA 91 (2015) 013847
1. Non-Gaussian states: vortex particles with orbital angular momentum (OAM)

Vortex electrons with $E = 300$ keV were generated in 2010:

- They can be focused to a spot of 0.1 нм
  

- Their OAM can be as high as 1000!
  

- Magnetic moment of such electrons is 3 orders of magnitude larger than the Bohr magneton!
  
  $K.Yu. \text{Bliokh, et al., PRL 107, 174802 (2011)}$
1. Non-Gaussian states: vortex particles with orbital angular momentum (OAM)

The huge magnetic moment $\rightarrow$ “Orbital light”:

Transition radiation:  
Angular asymmetry of $\sim 0.1 – 1\%$


FIG. 2 (color online). Distribution of the forward TR over $\theta_2$ for $\ell = 0$ (black solid line), $\ell = 1000$ (red dashed line), and $\ell = 10000$ (blue dotted line). Parameters are $\alpha = 70^\circ$, $\theta_1 = -40^\circ$, $h\omega = 5$ eV.
1. Non-Gaussian states: Airy beams

Berry, Balazs 1979

For an ideal Airy beam:

1. There is no spreading
2. Curved path in a free space
3. Self-healing after scattering

Experimental realization for 200 keV electrons →

2. Non-paraxial wave packets and the Wigner functions

We need a Lorentz-invariant description of the non-Gaussian wave packets beyond the paraxial regime!

A Gaussian packet of a massive boson:

\[ \psi(p) = \frac{2^{3/2}\pi}{\sigma} \frac{e^{-m^2/\sigma^2}}{\sqrt{K_1(2m^2/\sigma^2)}} \exp \left\{ \frac{(p - \bar{p})^2}{2\sigma^2} \right\} \]

\[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon} |\psi(p)|^2 = 1, \]

\[ p^2 = \bar{p}^2 = m^2 \]

In the paraxial regime this turns into a customary Gaussian packet:

\[ \frac{(p - \bar{p})^2}{2\sigma^2} = -\frac{1}{2\sigma^2} (\delta_{ij} - \bar{u}_i\bar{u}_j) (p - \bar{p})_i(p - \bar{p})_j + \mathcal{O}((p - \bar{p})^3) \]

Mean energy:

\[ \langle \varepsilon \rangle = \int d^3x \, T^{00} = \bar{\varepsilon} \frac{K_2 (2m^2/\sigma^2)}{K_1 (2m^2/\sigma^2)} = \bar{\varepsilon} \left( 1 + \frac{3}{4} \frac{\sigma^2}{m^2} + \mathcal{O}(\sigma^4/m^4) \right) \]

Non-paraxial correction!
2. Non-paraxial wave packets and the Wigner functions

A relativistic generalization for a vortex boson will be:

\[ \psi_\ell(p) = \frac{2^{3/2} \pi}{\sigma^{\ell+1} \sqrt{\ell!}} p^\ell \sqrt{K_{\ell+1}(2m^2/\sigma^2)} e^{-m^2/\sigma^2} \exp \left\{ \frac{(p - \bar{p})^2}{2\sigma^2} + i\ell \varphi_p \right\} \]

They are orthogonal:

\[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon} [\psi_\ell(p)]^* \psi_{\ell'}(p) = \delta_{\ell,\ell'} \]

An exact solution to the Klein-Gordon equation:

\[ \psi_\ell(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon} \psi_\ell(p) e^{-ipx} = \frac{(i\rho)^\ell}{\sqrt{2\ell!} \pi} \frac{\sigma^{\ell+1}}{s^{\ell+1}} \frac{K_{\ell+1}(\varsigma m^2/\sigma^2)}{\sqrt{K_{\ell+1}(2m^2/\sigma^2)}} e^{i\ell \varphi_r} \]

\[ \varsigma = \frac{1}{m} \sqrt{(\bar{p}_\mu + ix_\mu \sigma^2)^2} = \text{inv}, \quad \text{Re} \varsigma > 0 \]

And analogously for a spinning particle...

D.K., ArXiv: 1803.09150; 1803.10166
2. Non-paraxial wave packets and the Wigner functions

The mean momentum of such a vortex packet is

$$\langle p_{\ell}^\mu \rangle = \{\langle \varepsilon_{\ell} \rangle, \langle p_{\ell} \rangle \} = \{\bar{\varepsilon}, \bar{p}\} \frac{K_{|\ell|+2} (2m^2/\sigma^2)}{K_{|\ell|+1} (2m^2/\sigma^2)} \simeq \{\bar{\varepsilon}, \bar{p}\} \left(1 + \left(\frac{3}{4} + \frac{|\ell|}{2}\right) \frac{\sigma^2}{m^2}\right)$$

An invariant mass of this packet:

$$m_{\ell}^2 = \langle p_{\ell} \rangle^2 \simeq m^2 \left(1 + \left(\frac{3}{2} + |\ell|\right) \frac{\sigma^2}{m^2}\right)$$

With modern technology

(at el. microscopes):

$$\frac{\delta m_{\ell}}{m_{\text{inv}}} \simeq \frac{\delta m_{\ell}}{m} \lesssim 10^{-3} \quad |\ell| \sim 10^3 \quad \sigma_\perp \gtrsim 0.1 \text{ nm}$$

Analogously for the vortex electron’s magnetic moment:

$$\mu_f = \frac{1}{2} \int d^3r \mathbf{r} \times \bar{\psi}_f(x) \gamma \psi_f(x) \simeq \frac{1}{2\bar{\varepsilon}} (\zeta + \hat{\zeta} \ell) \left(1 + \mathcal{O}(|\ell|\sigma^2/m^2)\right)$$

Enhancement due to the OAM!

D.K., ArXiv: 1803.09150; 1803.10166
For scattering of wave packets instead of plane waves:

\[ S_{fi} = \langle pw | \hat{S} | i \rangle = \int \prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3} \psi_i(p_i) S_{fi}^{(pw)} \]

Is there a small parameter?

The plane-wave limit: \( \sigma_i \to 0, \quad p_i \to p_i' \to \langle p_i \rangle \) therefore \( \frac{p_i + p_i'}{2} \to \langle p_i \rangle, \quad p_i - p_i' \to 0 \)

In the new variables \( \frac{p_i + p_i'}{2} \to p_i, \quad p_i - p_i' \to k_i \) we get \( |k_i| \ll |p_i| \) when \( \sigma \ll m \)

A density matrix in these new variables is called a **Wigner function**!

Wigner 1932
2. Non-paraxial wave packets and the Wigner functions

The scattering probability can be expressed via the Wigner functions:

$$dW = |S_{fi}|^2 \prod_{f=3}^{N_f+2} V \frac{d^3p_f}{(2\pi)^3} = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \ d\sigma(k, p_{1,2}) \mathcal{L}^{(2)}(k, p_{1,2}),$$

*Kotkin, Serbo, Schiller, Int. J. Mod. Phys. A7 (1992) 4707*

$$d\sigma(k, p_{1,2}) = (2\pi)^4 \delta\left(\varepsilon_1(p_1 + k/2) + \varepsilon_2(p_2 - k/2) - \varepsilon_f\right) \delta^{(3)}(p_1 + p_2 - p_f)$$

$$\times T_{fi}^{(pu)}(p_1 + k/2, p_2 - k/2) T_{fi}^{*(pu)}(p_1 - k/2, p_2 + k/2) \frac{1}{v(p_1, p_2)} \prod_{f=3}^{N_f+2} \frac{d^3p_f}{(2\pi)^3},$$

Matches the customary cross section when $k = 0!$

$$\mathcal{L}^{(2)}(k, p_{1,2}) = v(p_1, p_2) \int dt d^3r d^3R e^{ikR} n_1(r, p_1, t)n_2(r + R, p_2, t),$$

$$v(p_1, p_2) = \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2} \over \varepsilon_1(p_1)\varepsilon_2(p_2) = \sqrt{(u_1 - u_2)^2 - [u_1 \times u_2]^2},$$

the Wigner functions
2. Non-paraxial wave packets and the Wigner functions

What do we lose in the paraxial regime?

For a non-relativistic Airy beam:

\[ \psi(p) = \pi^{3/4} \left( \frac{2}{\sigma} \right)^{3/2} \exp \left\{ -ir_0p - \frac{(p - \langle p \rangle)^2}{2\sigma^2} + \frac{i}{3} \left( \xi_x p_x^3 + \xi_y p_y^3 \right) \right\} \]

The exact Wigner function is (not everywhere positive)

\[
n(r, p, t; \xi) = 2^{13/3} \frac{\pi}{\sigma^2 \xi_x \xi_y} \exp \left\{ -\sigma^2(z - \langle z \rangle)^2 - \frac{(p - \langle p \rangle)^2}{\sigma^2} + \frac{1}{\sigma^2 \xi_x^3} \left( x - \langle x \rangle + \xi_x p_x^2 + \frac{1}{6\sigma^4 \xi_x^3} \right) + \frac{1}{\sigma^2 \xi_y^3} \left( y - \langle y \rangle + \xi_y p_y^2 + \frac{1}{6\sigma^4 \xi_y^3} \right) \right\} \times \text{Ai} \left[ \frac{2^{2/3}}{\xi_x} \left( x - \langle x \rangle + \xi_x p_x^2 + \frac{1}{4\sigma^4 \xi_x^3} \right) \right] \text{Ai} \left[ \frac{2^{2/3}}{\xi_y} \left( y - \langle y \rangle + \xi_y p_y^2 + \frac{1}{4\sigma^4 \xi_y^3} \right) \right]
\]

The approximate/paraxial one is (everywhere positive)

\[ n(r, p, t; \xi) = 8 \exp \left\{ -\frac{(p - \langle p \rangle)^2}{\sigma^2} - \sigma^2 (r - \langle r \rangle + \eta)^2 \right\} \]

\[ \eta \equiv \eta(p) = \{ \xi_x^3 p_x^2, \xi_y^3 p_y^2, 0 \} \]

Possible quantum decoherence is lost!
2. Non-paraxial wave packets and the Wigner functions

\[ m = 1, \sigma / \langle p \rangle_z = 1/5, \xi_x = \xi_y = 2/\sigma, r_0 = z = t = \langle p \rangle_\perp = 0 \]
3. Non-paraxial effects in scattering

We represent the scattering amplitude as follows:

\[ T_{fi} = |T_{fi}| \exp\{i\zeta_{fi}\} \]

\[ T_{fi}(p_1 + k/2, p_2 - k/2)T_{fi}^*(p_1 - k/2, p_2 + k/2) \approx \]

\[ \approx \left( |T_{fi}|^2 + \frac{1}{4} k_i k_j C_{ij} + O(k^4) \right) \exp \left\{ i k \partial_{\Delta p} \zeta_{fi} + O(k^3) \right\} \]

\[ \partial_{\Delta p} = \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}, \]

\[ C_{ij}(p_1, p_2) = |T_{fi}| \partial_{\Delta p_i} \partial_{\Delta p_j} |T_{fi}| - (\partial_{\Delta p_i} |T_{fi}|)(\partial_{\Delta p_j} |T_{fi}|) \]

\[ \tilde{b}_\varphi = b - \frac{\partial \varphi_1(p_1)}{\partial p_1} + \frac{\partial \varphi_2(p_2)}{\partial p_2} - \left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) \zeta_{fi}. \]

The amplitude’s phase

Impact-parameter

Phases of the in-states
3. Non-paraxial effects in scattering

We derive the first correction to the plane-wave cross section:

\[ d\sigma = dN/L \approx d\sigma^{(pw)} + d\sigma^{(1)} \]

\[ d\sigma^{(pw)} = \frac{dN^{(pw)}}{L^{(pw)}} = N_{b,1}N_{b,2}(2\pi)^4\delta^{(4)}(\langle p \rangle_1 + \langle p \rangle_2 - p_f)\frac{|T_{fi}|^2}{v} \prod_{f=3}^{N_f+2} \frac{d^3p_f}{(2\pi)^3} \]

provided the packets do not spread much during the collision: \( t_{\text{col}} \ll t_{\text{diff}} \sim \frac{\sigma_b}{u_\perp} \sim \sigma_b^2 \varepsilon \)

\[ \frac{d\sigma^{(1)}}{d\sigma^{(pw)}} = \text{“geometric” terms} + \text{dynamic terms} \]

\[ \sim \frac{\sigma_1^2}{m_1^2} \quad \text{and} \quad \sim \frac{\sigma_2^2}{m_2^2} \]

Also depend on the phases and on an overlap of the in-states

D.K., JHEP 03 (2017) 049
3. Non-paraxial effects in scattering

To be more precise:

\[
\begin{align*}
 dW &= \prod_{f=3}^{N_f+2} \frac{d^3p_f}{(2\pi)^3} \frac{(2\pi)^{11}}{(\pi^3\sigma_1\sigma_2)^3} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \\
&\times \delta(\varepsilon_1(p_1 - k/2) + \varepsilon_2(p_2 + k/2) - \varepsilon_f) \\
&\times \delta(\varepsilon_1(p_1 + k/2) + \varepsilon_2(p_2 - k/2) - \varepsilon_f) \delta(p_1 + p_2 - p_f) \\
&\times T_{fi}(p_1 + k/2, p_2 - k/2) T_{fi}^*(p_1 - k/2, p_2 + k/2) \\
&\times \exp \left\{ -\frac{(p_1 - \langle p \rangle_1)^2}{\sigma_1^2} - \frac{(p_2 - \langle p \rangle_2)^2}{\sigma_2^2} - k^2 \left( \frac{1}{(2\sigma_1)^2} + \frac{1}{(2\sigma_2)^2} \right) - \\
&- i\kappa b + i(\varphi_1(p_1 + k/2) - \varphi_1(p_1 - k/2) + \varphi_2(p_2 - k/2) - \varphi_2(p_2 + k/2)) \right\}
\end{align*}
\]

\[
\mathcal{L}(p_1, p_2, k) = \frac{(2\pi)^7 \nu}{(\pi \sigma_1 \sigma_2)^3} \delta(\varepsilon_1(p_1 + k/2) - \varepsilon_1(p_1 - k/2) + \varepsilon_2(p_2 - k/2) - \varepsilon_2(p_2 + k/2)) \\
\times \exp \left\{ -\frac{(p_1 - \langle p \rangle_1)^2}{\sigma_1^2} - \frac{(p_2 - \langle p \rangle_2)^2}{\sigma_2^2} - k^2 \left( \frac{1}{(2\sigma_1)^2} + \frac{1}{(2\sigma_2)^2} \right) - \\
- i\kappa b + i(\varphi_1(p_1 + k/2) - \varphi_1(p_1 - k/2) + \varphi_2(p_2 - k/2) - \varphi_2(p_2 + k/2)) \right\}.
\]

\[
\frac{d\sigma^{(1)}}{d\sigma^{(pw)}} = -\frac{1}{4} \left( \frac{\text{Tr}\sigma_1^2 - 3u_1\sigma_1^2u_1}{\varepsilon_1^2} + \frac{\sqrt{\Delta u}\alpha^{-1}\Delta u}{v} \partial_{p_1}\sigma_1^2 \partial_{p_1} \frac{v}{\sqrt{\Delta u}\alpha^{-1}\Delta u} + (1 \rightarrow 2) \right) - \\
- \frac{1}{2} \left( \frac{\sqrt{\Delta u}\alpha^{-1}\Delta u}{|T_{fi}|^2} (\partial_{\varepsilon_1} + \partial_{\varepsilon_2}) \left( \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) \left( \text{Tr}\alpha_\perp^{-1} - \frac{1}{2} u_1\alpha_\perp^{-1}u_1 - \frac{1}{2} u_2\alpha_\perp^{-1}u_2 \right) \right. \\
\times \frac{|T_{fi}|^2}{\sqrt{\Delta u}\alpha^{-1}\Delta u} - \frac{1}{2|T_{fi}|^2} C_{ij} \alpha_{\perp ij}^{-1} - 2\langle b_\varphi \rangle\alpha_{\perp}^{-1} \partial_{\Delta p} \zeta_{fi} + \partial_{\Delta p} \zeta_{fi} \alpha_{\perp}^{-1} \partial_{\Delta p} \zeta_{fi} \right) 
\]
3. Non-paraxial effects in scattering

Interference of the incoming packets is governed by

\[
\left( \frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} \right)^{-1} = \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}
\]

Due to the finite overlap of the two non-orthogonal packets!

A corresponding term in the cross section is:

\[
\frac{d\sigma^{(1)}}{d\sigma^{(pw)}} \propto \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \left[ \frac{\Delta u}{|\Delta u|} \times \left[ \frac{\Delta u}{|\Delta u|} \times \langle b_\varphi \rangle \right] \right] \cdot \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \right) \zeta_{fi} \bigg|_{p_{1,2} = \langle p \rangle_{1,2}}
\]

\[
\Delta u = u_1 - u_2,
\]

\[
b_\varphi = b - \frac{\partial \varphi_1(p_1)}{\partial p_1} + \frac{\partial \varphi_2(p_2)}{\partial p_2}
\]

This contribution defines an azimuthal asymmetry of the scattering:

\[
A[b_\varphi] = \frac{dW[b_\varphi] - dW[-b_\varphi]}{dW[b_\varphi] + dW[-b_\varphi]} = \frac{d\sigma[b_\varphi] - d\sigma[-b_\varphi]}{d\sigma[b_\varphi] + d\sigma[-b_\varphi]} = \frac{d\sigma^{(1)}[b_\varphi] - d\sigma^{(1)}[-b_\varphi]}{2d\sigma^{(pw)}} + \mathcal{O}(\sigma^4)
\]

D.K., JHEP 03 (2017) 049
3. Non-paraxial effects in scattering

\[ A = \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \left[ \frac{\Delta u}{|\Delta u|} \times \left[ \frac{\Delta u}{|\Delta u|} \times \langle b_\varphi \rangle \right] \right] \cdot \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \right) \zeta_{fi} \bigg|_{p_{1,2} = \langle p \rangle_{1,2}} \]

There are two scenarios:

1. **Off-center collision** of the Gaussian beams

2. Central collision of **non-Gaussian** beams (vortex particles, Airy beams, etc.)

For a \(1 + 2 \rightarrow 3 + 4\) process in the collider frame:

\[ \langle p \rangle_1 = -\langle p \rangle_2 \equiv p = u\varepsilon = \{0, 0, p\}, \ \Delta u = 2u, \ v = |\Delta u| \]

\[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} = 8p \frac{\partial}{\partial s} + 4(p_3 - p) \frac{\partial}{\partial t} \]

\[ t = (p_1 - p_3)^2, \ s = (p_1 + p_2)^2 \]

and the asymmetry simplifies:

\[ A = 4\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \langle b_\varphi \rangle_{p_3} \frac{\partial \zeta_{fi}(s, t)}{\partial t} \]

It is odd with respect to \( \phi_3 \rightarrow \phi_3 \pm \pi \)

An up-down asymmetry!

Shows how the phase changes with the transferred momentum!
3. Non-paraxial effects in scattering

The 1\textsuperscript{st} scenario: non-central collision of Gaussian beams with $b \lesssim \sigma_b$
(identical beams, relativistic energies, small scattering angles) – say, ee $\to$ X, pp $\to$ X, etc.

\[ A \approx -2 \frac{\lambda_c}{\sigma_b} \cos \phi_{sc} \sqrt{\tau_0} \frac{\partial \zeta_{fi}}{\partial \tau_0}, \quad \tau_0 = \frac{-t}{4m^2} \]

or, alternatively:

\[ A \approx -2 \frac{\lambda_c}{\sigma_b} \cos \phi_{sc} \frac{1}{\gamma} \frac{\partial \zeta_{fi}}{\partial \theta_{sc}} \]

Just a linear “geometric” suppression!

In other words: $d\sigma^{(1)} \propto f(s, t) \frac{\lambda_c^2}{\sigma_b^2}$ and there is a region where $f(s, t)$ is very large!

In QED (West, Yennie, 1968):

\[ \frac{1}{\gamma} \frac{\partial \zeta_{fi}}{\partial \theta_{sc}} \approx \frac{\alpha_{em}}{\gamma \theta_{sc}} \]

\[ A = \mathcal{O}\left(\frac{\lambda_c \alpha_{em}}{\sigma_b \gamma \theta_{sc}}\right) \]

Lorentz invariant!

For electrons of $E = 300$ keV focused to $0.1$ nm
and $\theta_{sc} \sim 10^{-2} - 10^{-1}$ we have the following conservative estimate:

\[ |A| \sim 10^{-4} - 10^{-3} \]

And the same estimate within the 2\textsuperscript{nd} scenario

One can detect a contribution of the Coulomb phase!

Similar estimates were also obtained by Ivanov 2012 and Ivanov, et al. 2016
3. Non-paraxial effects in scattering

A parameter which is usually employed: \[ \rho = \frac{\text{Re} T_{fi}}{\text{Im} T_{fi}} = \frac{1}{\tan \zeta_{fi}} \]

Once the Coulomb phase is known, one can retrieve also the hadronic phase!

Measurement of elastic pp scattering at \( \sqrt{s} = 8 \text{ TeV} \) in the Coulomb–nuclear interference region: determination of the \( \rho \)-parameter and the total cross-section

TOTEM Collaboration

\[ |t|, \text{ from } 6 \times 10^{-4} \text{ to } 0.2 \text{ GeV}^2 \]

The region of a Coulomb-hadronic interference
3. Non-paraxial effects in scattering

Taking the same models as TOTEM, one can give more precise estimates of the asymmetry induced by the hadronic phase:

\[
\frac{\partial \zeta_{fi}}{\partial t} = -\frac{\tau}{\tau^2 + (t + |t_0|)^2} - \text{the so-called standard parametrization, red dotted line}
\]

\[
\frac{\partial \zeta_{fi}}{\partial t} = -\frac{\rho t_d}{(\rho t_d)^2 + (t - t_d)^2} - \text{the one by Bailly et al. (EHS-RCBC Collaboration), Z. Phys. C 37, 7 (1987) blue solid line}
\]

\[
\frac{\partial \zeta_{fi}}{\partial t} = \zeta_1 (\kappa + \nu t) \left( \frac{-t}{1 \text{GeV}^2} \right)^{\kappa-1} e^{\nu t} - \text{the so-called peripheral parametrization V. Kundrát and M. Lokajíček, Z. Phys. C 63, 619 (1994) the green dashed line}
\]

For pp-collisions the beams are too wide...

\[
\frac{\lambda_c}{\sigma_b} \sim 10^{-11}.
\]
3. Non-paraxial effects in scattering

When the parameter $\lambda_c/\sigma_b$ is small, the quantum decoherence does not reveal itself in scattering and the Wigner functions stay everywhere positive (the WKB approximation).

Is there a chance to probe negative values of a Wigner function in scattering?

Beam-beam collision $\rightarrow$ beam + target

For scattering of an electron off an atom, an analogous small parameter is

$$\frac{a}{\sigma_b} \quad a \approx 0.053 \text{ nm is a Bohr radius}$$

which is 137 times larger than $\lambda_c/\sigma_b$!

For electron beams focused to 0.1 nm one can enter the non-paraxial regime!
3. Non-paraxial effects in scattering

The so-called Schrödinger’s cat state $|r_0\rangle \pm |-r_0\rangle$
has a not-everywhere positive Wigner function

$r_0$ is an impact parameter
3. Non-paraxial effects in scattering

In the Born approximation, the number of scattering events is:

\[
\frac{d\nu}{d\Omega} = N_e \int d^2b \, d^2p \, n(b) \, W(b, p) (f(Q-p))^2.
\]

The target’s transverse profile  The projectile’s Wigner function  The Born amplitude

For a wide Gaussian target of hydrogen in the ground 1s state:

\[
\frac{d\nu_{1\pm1}}{d\Omega} = N_{1\pm1} \int_0^\infty dx \, e^{-xg} \, \frac{x + x^2 + x^3/6}{1 + xa^2/(8\sigma^2)} \left( \cosh \left( \frac{b_0 \cdot r_0}{\Sigma^2} \right) e^{-r_0^2/(2\Sigma^2)} \right) \\
\pm \cos \left( 2r_0 \cdot p_f \frac{xa^2/(8\sigma^2)}{1 + xa^2/(8\sigma^2)} \right) \exp \left\{ - \frac{r_0^2}{2\Sigma^2(1 + xa^2/(8\sigma^2))} \right\}
\]

Quantum interference does contribute to the cross section already in the Born approximation!

D.K., V.G. Serbo, PRL 119 (2017) 173601
3. Non-paraxial effects in scattering

The quantum interference also results in an angular asymmetry:

\[ \sigma_\perp = 2a \approx 0.1 \text{ nm (FWHM} \approx 0.25 \text{ nm) \]
\[ p_i = p_f = 20/a \ (\varepsilon_{\text{kin}} = 5.6 \text{ keV) \]
\[ \theta = 10^\circ \]

Several per cent!

D.K., V.G. Serbo, PRL 119 (2017) 173601
Conclusion

- The non-paraxial effects are effectively attenuated by $\lambda_c/\sigma_b$, not always by $\lambda_c^2/\sigma_b^2$, and originate thanks to a finite overlap of the incoming wave packets.
- For instance, for vortex electrons with high OAM, $d\sigma^{(1)}/d\sigma_{pw} \sim |\ell| \lambda_c^2/\sigma_b^2$.
- In beam-beam collisions, the Wigner functions of the non-Gaussian in-states turn out to be everywhere positive.
- For well-focused electron beams of different spatial profiles these effects can already reach $\sim 0.1 – 1\%$.
- For QED, they can compete with the NNLO- or even with the NLO corrections.
- A contribution of the Coulomb phase to the cross section in elastic $ee$ scattering can reach $\sim 0.1\%$ and can already be measured.
- The quantum decoherence (connected with the Wigner functions' negativity) may reveal itself in scattering off an atomic target already in the Born approximation; The corresponding effects can also reach $1 – 10\%$ for the Schrödinger’s cat states.
The steps further and issues:

- Proton spin puzzle: vortex beams can be more sensitive to the partons' angular momenta.

  Does a deep inelastic $e^{tw} + p \rightarrow X$ scattering bring some new information?

- Are there effects that are enhanced when ultrarelativistic massive particles with a non-Gaussian profile collide or are scattered off a target?

  How can one use ultrarelativistic vortex electrons?

- Can we employ scattering/annihilation as a means for quantum tomography?

- ....