# On the relation between pole and running masses of heavy quarks and charged leptons

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# Pole masses of heavy quarks

The full bare quark propagator has the following form:

$$\hat{\mathrm{S}}(k) = \frac{i}{\hat{k} - m_{0,q} - \Sigma(\hat{k})} \ ,$$

where $\hat{\Sigma}$ is the one-particle irreducible fermion self-energy operator, $m_{0,q}$ is the unrenormalized bare mass of the $q$-th quark.

$$\Sigma(\hat{k}) = m_{0,q}\Sigma_1(k^2) + (\hat{k} - m_{0,q})\Sigma_2(k^2)$$

From the on-shell mass condition $(\hat{k} - m_{0,q} - \Sigma(\hat{k}))\Big|_{k^2=M_q^2} = 0$

one can find:

$$M_q = m_{0,q}(1 + \Sigma_1(M_q^2)) \ .$$

Thus this allows to obtain the relation between bare and pole mass of heavy quarks:

$$m_{0,q} = Z_m^{\mathrm{OS}} \ M_q \ .$$

# Running heavy quark mass

Similarly, the analogous relation between bare and running mass in the $\overline{\rm MS}$-scheme can be written as:

$$m_{0,q} = Z_m^{\overline{\rm MS}}\, \overline{m}_q(\mu^2)\ ,$$

with scale parameter $\mu$, appearing in the framework of dimensional regularization.
Further we define the following RG-quantities:

$$\beta(\alpha_s) = \mu^2 \frac{\partial}{\partial \mu^2}\left(\frac{\alpha_s(\mu^2)}{\pi}\right) = -\sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s}{\pi}\right)^{i+2},$$

$$\gamma_m(\alpha_s) = \mu^2 \frac{\partial}{\partial \mu^2} \log \overline{m}_q(\mu^2) = -\sum_{i=0}^{\infty} \gamma_i \left(\frac{\alpha_s}{\pi}\right)^{i+1},$$

where $\beta(\alpha_s)$ and $\gamma_m(\alpha_s)$ are calculated at present in analytical form at the 5-loop order in the $\overline{\rm MS}$-scheme.

# Running heavy quark mass

The evolution of the running mass is described by the following equation ($\alpha_s/\pi = a_s$):

$$\frac{\overline{m}_q(\tilde{\mu}^2)}{\overline{m}_q(\mu^2)} = \exp\left(\int\limits_{a_s(\mu^2)}^{a_s(\tilde{\mu}^2)} dx\, \frac{\gamma_m(x)}{\beta(x)}\right) = 1 + \sum_{n=1}^{6} b_n a_s^n(\mu^2)\ ,$$

$$b_1 = \gamma_0 l\ , \qquad b_2 = \frac{\gamma_0}{2}(\beta_0+\gamma_0)l^2 + \gamma_1 l\ ,$$

$$\begin{aligned}
b_3 &= \frac{\gamma_0}{3}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)l^3 + \left(\beta_1\frac{\gamma_0}{2} + \gamma_1\beta_0 + \gamma_1\gamma_0\right)l^2 + \gamma_2 l\ ,\\
b_4 &= \frac{\gamma_0}{4}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\left(\beta_0+\frac{\gamma_0}{3}\right)l^4 + \left(\frac{5}{6}\beta_1\beta_0\gamma_0 + \frac{\beta_1\gamma_0^2}{2}\right.\\
&+ \left.\gamma_1(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\right)l^3 + \left(\beta_2\frac{\gamma_0}{2} + \gamma_1\beta_1 + \frac{\gamma_1^2}{2} + \frac{3}{2}\gamma_2\beta_0 + \gamma_2\gamma_0\right)l^2\\
&+ \gamma_3 l\ , \qquad \text{where } l = \log(\mu^2/\tilde{\mu}^2),
\end{aligned}$$

# Running heavy quark mass

$$
\begin{aligned}
b_5 \quad = \quad & \frac{\gamma_0}{5}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\left(\beta_0+\frac{\gamma_0}{3}\right)\left(\beta_0+\frac{\gamma_0}{4}\right)l^5+\left(\gamma_1\beta_0^3+\frac{13}{12}\gamma_0\beta_1\beta_0^2\right.\\
+ \quad & \left.\frac{13}{12}\gamma_0^2\beta_1\beta_0+\frac{11}{6}\gamma_0\gamma_1\beta_0^2+\gamma_0^2\gamma_1\beta_0+\frac{1}{4}\beta_1\gamma_0^3+\frac{1}{6}\gamma_1\gamma_0^3\right)l^4\\
+ \quad & \left(\gamma_0\beta_2\beta_0+2\gamma_0\beta_0\gamma_2+\frac{7}{3}\gamma_1\beta_1\beta_0+\frac{3}{2}\gamma_0\gamma_1\beta_1+\frac{1}{2}\gamma_0\beta_1^2+2\beta_0^2\gamma_2+\beta_0\gamma_1^2\right.\\
+ \quad & \left.\frac{1}{2}\beta_2\gamma_0^2+\frac{1}{2}\gamma_0\gamma_1^2+\frac{1}{2}\gamma_2\gamma_0^2\right)l^3+\left(\frac{1}{2}\gamma_0\beta_3+\gamma_1\beta_2+\frac{3}{2}\beta_1\gamma_2+2\beta_0\gamma_3\right.\\
+ \quad & \left.\gamma_1\gamma_2+\gamma_0\gamma_3\right)l^2+\gamma_4 l\ ,
\end{aligned}
$$

where four-loop terms $\beta_3$ and $\gamma_3$ were calculated by (*Ritbergen, Vermaseren, Larin, 1997*) and five-loop coefficient $\gamma_4$ was computed by (*Baikov, Chetyrkin, Kühn, 2014*).

# Running heavy quark mass

$$
\begin{aligned}
b_6 &= \frac{\gamma_0}{6}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\left(\beta_0+\frac{\gamma_0}{3}\right)\left(\beta_0+\frac{\gamma_0}{4}\right)\left(\beta_0+\frac{\gamma_0}{5}\right)l^6 \\
&+ \left(\frac{1}{12}\beta_1\gamma_0^4+\gamma_1\beta_0^4+\frac{1}{24}\gamma_1\gamma_0^4+\frac{5}{3}\beta_0^2\beta_1\gamma_0^2+\frac{35}{24}\beta_0^2\gamma_0^2\gamma_1+\frac{2}{3}\beta_0\beta_1\gamma_0^3\right. \\
&+ \left.\frac{77}{60}\beta_0^3\beta_1\gamma_0+\frac{5}{12}\beta_0\gamma_0^3\gamma_1+\frac{25}{12}\beta_0^3\gamma_0\gamma_1\right)l^5+\left(\frac{1}{4}\beta_2\gamma_0^3+\frac{5}{2}\beta_0^3\gamma_2+\frac{1}{6}\gamma_0^3\gamma_2\right. \\
&+ \frac{3}{2}\beta_0^2\gamma_1^2+\frac{5}{8}\beta_1^2\gamma_0^2+\frac{1}{4}\gamma_0^2\gamma_1^2+\frac{35}{24}\beta_0\beta_1^2\gamma_0+\frac{5}{4}\beta_0\beta_2\gamma_0^2+\frac{47}{12}\beta_0^2\beta_1\gamma_1 \\
&+ \left.\frac{3}{2}\beta_0^2\beta_2\gamma_0+\frac{5}{4}\beta_0\gamma_0\gamma_1^2+\frac{5}{4}\beta_0\gamma_0^2\gamma_2+\beta_1\gamma_0^2\gamma_1+\frac{37}{12}\beta_0^2\gamma_0\gamma_2+\frac{25}{6}\beta_0\beta_1\gamma_0\gamma_1\right)l^4 \\
&+ \left(\beta_1\gamma_1^2+\frac{4}{3}\beta_1^2\gamma_1+\frac{1}{2}\beta_3\gamma_0^2+\frac{10}{3}\beta_0^2\gamma_3+\frac{1}{2}\gamma_0^2\gamma_3+\frac{1}{6}\gamma_1^3+\frac{9}{2}\beta_0\beta_1\gamma_2+\frac{8}{3}\beta_0\beta_2\gamma_1\right. \\
&+ \frac{7}{6}\beta_0\beta_3\gamma_0+\frac{7}{6}\beta_1\beta_2\gamma_0+\frac{5}{2}\beta_0\gamma_0\gamma_3+\frac{5}{2}\beta_0\gamma_1\gamma_2+2\beta_1\gamma_0\gamma_2+\frac{3}{2}\beta_2\gamma_0\gamma_1 \\
&+ \left.\gamma_0\gamma_1\gamma_2\right)l^3+\left(\frac{1}{2}\gamma_2^2+\frac{3}{2}\beta_2\gamma_2+\frac{5}{2}\beta_0\gamma_4+2\beta_1\gamma_3+\beta_3\gamma_1+\frac{1}{2}\beta_4\gamma_0+\gamma_0\gamma_4\right. \\
&+ \left.\gamma_1\gamma_3\right)l^2+\gamma_5 l\ ,
\end{aligned}
$$

where $\beta_4$ is known by (*Baikov, Chetyrkin, Kühn, 2017*).

# $\overline{\text{MS}}$-on-shell mass relation

Now define the $z_m$-ratio:

$$z_m(\mu^2) = \frac{\overline{m}_q(\mu^2)}{M_q} = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} = 1 + \sum_{i=1}^{\infty} z_m^{(i)} a_s^i(\mu^2) \ .$$

Wherein, all renormalized coupling constants are expressed in terms of a single constant (in the $\overline{\text{MS}}$-scheme).

$$\alpha_{s,\,0} = \mu^{2\varepsilon} \exp\left( - \int\limits_0^{\alpha_s} \frac{dx}{x} \frac{\beta(x)}{\beta(x) - \varepsilon x} \right) =$$

$$= \mu^{2\varepsilon} \alpha_s \left( 1 - \frac{\beta_0}{\varepsilon} a_s + \left( \frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon} \right) a_s^2 - \left( \frac{\beta_0^3}{\varepsilon^3} - \frac{7\beta_1\beta_0}{6\varepsilon^2} + \frac{\beta_2}{3\varepsilon} \right) a_s^3 + \ldots \right) \ .$$

# $\overline{\text{MS}}$-on-shell mass relation

Coefficients $z_m^{(i)}$ with $1 \leqslant i \leqslant 3$ are calculated in analytical form for gauge color SU($N_c$)-group. For case of the SU$_c$(3)-group with Casimir operator $C_F = 4/3$, $C_A = 3$ $((t^a t^a)_{ij} = C_F \delta_{ij},\ f^{acd} f^{bcd} = C_A \delta^{ab})$ at the renormalization point $\mu^2 = M_q^2$:

$$z_m^{(1)} = -\frac{4}{3}\ , \quad (\textit{Tarrach, 1981})$$

$$z_m^{(2)} = -14.3323 + 1.04136 n_l\ , \quad (\textit{Gray, Broadhurst ... 1990})$$

$$z_m^{(3)} = -198.706 + 26.9239 n_l - 0.65269 n_l^2 \quad (\textit{Melnikov, Ritbergen, 2000})$$

and independently (*Chetyrkin, Steinhauser, 2000*).

We define $n_l = n_f - 1$, $n_l$ is the number of massless quarks. Analytic expression for the $z_m^{(3)}$-term contains not only Riemann zeta-functions $\zeta_n = \sum_{k=1}^{\infty} k^{-n}$ up to $n = 5$, but also polylogarithmic function $\text{Li}_n(x) = \sum_{k=1}^{\infty} x^k k^{-n}$ with $n = 4$ and 5 at $x = 1/2$.

# $z_m^{(4)}$-coefficient

Separately consider the four-loop term $z_m^{(4)}$. Like any-order term $z_m^{(i)}$, it can be expanded in powers of $n_l$:

$$z_m^{(4)} = z_m^{(40)} + z_m^{(41)} n_l + z_m^{(42)} n_l^2 + z_m^{(43)} n_l^3 \ .$$

In this expression the last two coefficients are known analytically *(Lee, Marquard, Smirnov A.V., Smirnov V. A., Steinhauser, 2013)*, and the first two, namely the constant contribution $z_m^{(40)}$ and the linear dependent on $n_l$ term $z_m^{(41)}$, are not yet computed analytically:

$$z_m^{(4)} = z_m^{(40)} + z_m^{(41)} n_l - 43.4824 n_l^2 + 0.67814 n_l^3 \ .$$

# $z_m^{(4)}$-coefficient

In the work of *(Marquard, Smirnov A., Smirnov V., Steinhauser, Wellmann, 2016)* the values of the four-loop correction $z_m^{(4)}$ were obtained at fixed number $n_l$ in the wide region $0 \leq n_l \leq 20$. To extract the unknown coefficients $z_m^{(40)}$ and $z_m^{(41)}$ we use the least squares method (LSM) as a method of solving the overdetermined system of equations. However, we propose to consider the Banks-Zaks ansatz-motivated values of $n_l$ only ($\beta_0(n_f) > 0$), namely $3 \leq n_l \leq 15$:

$$\begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \\ 1 & 14 \\ 1 & 15 \end{pmatrix} \begin{pmatrix} z_m^{(40)} \\ \\ \\ z_m^{(41)} \end{pmatrix} = \begin{pmatrix} -1383.33 \pm 1.74 \\ -626.38 \pm 1.77 \\ 130.56 \pm 1.80 \\ 887.50 \pm 1.84 \\ 1644.45 \pm 1.87 \\ 2401.39 \pm 1.91 \\ 3158.33 \pm 1.94 \\ 3915.27 \pm 1.98 \\ 4672.22 \pm 2.01 \\ 5429.15 \pm 2.05 \\ 6186.09 \pm 2.08 \\ 6943.03 \pm 2.12 \\ 7699.98 \pm 2.16 \end{pmatrix}$$

# Application of the least squares method

We introduce the $\Phi$-function, which is equal to the sum of the squares of the deviations of all equations in the system. Under the solution we mean such values $z_m^{(40)}$ and $z_m^{(41)}$ for which the $\Phi$-function has the minimum:

$$\Phi(z_m^{(40)}, z_m^{(41)}) = \sum_{k=1}^{13} \Delta_k^2 = \sum_{k=1}^{21} (z_m^{(40)} + z_m^{(41)} n_{l_k} - y_{l_k})^2$$

$$\frac{\partial \Phi}{\partial z_m^{(40)}} = 0, \quad \frac{\partial \Phi}{\partial z_m^{(41)}} = 0.$$

$$\Delta z_m^{(40)} = \frac{1}{13 \sum\limits_{k=1}^{13} n_{l_k}^2 - \left( \sum\limits_{k=1}^{13} n_{l_k} \right)^2} \sqrt{\sum_{k=1}^{13} \Delta y_{l_k}^2 \left( \sum_{i=1}^{13} n_{l_i}^2 - n_{l_k} \sum_{i=1}^{13} n_{l_i} \right)^2}$$

$$\Delta z_m^{(41)} = \frac{1}{13 \sum\limits_{k=1}^{13} n_{l_k}^2 - \left( \sum\limits_{k=1}^{13} n_{l_k} \right)^2} \sqrt{\sum_{k=1}^{13} \Delta y_{l_k}^2 \left( 13 n_{l_k} - \sum_{i=1}^{13} n_{l_i} \right)^2}$$

# Numerical results for $z_m^{(40)}$ and $z_m^{(41)}$-terms and their uncertainties

The LSM allows us to obtain the following values:

$$z_m^{(40)}(M_q^2) = -3654.14 \pm 1.34, \quad z_m^{(41)}(M_q^2) = 756.94 \pm 0.15.$$

which agrees with the results of *(Marquard, Smirnov A. ... 2016)* for $z_m^{(40)}$-term, obtained without considering the correlation of equations of the overdetermined system at $n_l = 0$:

$$z_m^{(40)}(M_q^2) = -3654.15 \pm 1.64, \quad z_m^{(41)}(M_q^2) = 756.942 \pm 0.040.$$

The previous values, obtained in the work of *(Kataev, Molokoedov, 2016)* with using three points only ($n_l = 3, 4, 5$ and taken from *(Marquard, Smirnov A., Smirnov V., Steinhauser, 2015)*), read:

$$z_m^{(40)}(M_q^2) = -3642.9 \pm 62.0, \quad z_m^{(41)}(M_q^2) = 757.05 \pm 15.20.$$

One can see that compared with the inaccuracies of the $z_m^{(40)}$ and $z_m^{(41)}$-terms their central values vary slightly. Thus:

$$\overline{m}_q(M_q^2) \approx M_q(1 - 1.33333a_s + (1.0414n_l - 14.332)a_s^2 +$$
$$+(-0.6527n_l^2 + 26.924n_l - 198.71)a_s^3 +$$
$$+(0.6781n_l^3 - 43.482n_l^2 + \underline{(756.94 \pm 0.15)}n_l \underline{-3654.14 \pm 1.34})a_s^4 + \mathcal{O}(a_s^5))$$

# Asymptotic structure

Shifting $\mu^2 = M_q^2 \to \overline{m}_q^2$ one can find the following expansions of the pole masses of $c, b$ and $t$-quarks ($\overline{a}_s = \alpha_s(\overline{m}_q^2)/\pi$):

$$
\begin{aligned}
M_c &\approx \overline{m}_c(\overline{m}_c^2)(1 + 1.3333\,\overline{a}_s + 10.318\,\overline{a}_s^2 + 116.49\,\overline{a}_s^3 + (1702.70 \pm 1.41)\,\overline{a}_s^4)\ , \\
M_b &\approx \overline{m}_b(\overline{m}_b^2)(1 + 1.3333\,\overline{a}_s + 9.277\,\overline{a}_s^2 + 94.41\,\overline{a}_s^3 + (1235.66 \pm 1.47)\,\overline{a}_s^4)\ , \\
M_t &\approx \overline{m}_t(\overline{m}_t^2)(1 + 1.3333\,\overline{a}_s + 8.236\,\overline{a}_s^2 + 73.63\,\overline{a}_s^3 + (839.14 \pm 1.54)\,\overline{a}_s^4)\ .
\end{aligned}
$$

These expressions demonstrate the property of the asymptotic structure of the perturbative QCD series. Indeed, one can see that all relations contain significantly growing and strictly sign-constant coefficients.
For numerical studies we use the average PDG(16) values of the running masses of $c$ and $b$-quarks, namely $\overline{m}_c(\overline{m}_c^2) = 1.280 \pm 0.030$ GeV, $\overline{m}_b(\overline{m}_b^2) = 4.180^{+0.040}_{-0.030}$ GeV. For top quark we assume $\overline{m}_t(\overline{m}_t^2) = 163.5$ GeV that does not contradict the data presented in PDG(16). As the initial normalization point we take $\alpha_s(M_Z^2) = 0.1182$ at $M_Z = 91.1876$ GeV.

# $N^3$LO numerical analysis

$$\Lambda_{\overline{\rm MS}}^{(n_l=3)} = 289 \text{ MeV} , \quad \alpha_s(\overline{m}_c^2) = 0.3818 ,$$
$$\Lambda_{\overline{\rm MS}}^{(n_l=4)} = 211 \text{ MeV} , \quad \alpha_s(\overline{m}_b^2) = 0.2252 ,$$
$$\Lambda_{\overline{\rm MS}}^{(n_l=5)} = 90 \text{ MeV} , \quad \alpha_s(\overline{m}_t^2) = 0.1087 .$$

$$\frac{M_c}{1 \text{ GeV}} \approx 1.28 + 0.207 + 0.195 + 0.268 + 0.475 \pm 0.040 ,$$
$$\frac{M_b}{1 \text{ GeV}} \approx 4.18 + 0.399 + 0.199 + 0.145 + 0.136^{+0.046}_{-0.034} ,$$
$$\frac{M_t}{1 \text{ GeV}} \approx 163.5 + 7.543 + 1.612 + 0.499 + 0.197 = 173.351 .$$

For $c$-quark pole mass its PT series has explicit asymptotic structure, beginning with 2-3 loop order. Therefore at these levels of PT one should use the concept of the running mass of $c$-quark. For $b$-quark it is possible to use the pole mass up to four-loop level. For $t$-quark at the $\mathcal{O}(a_s^4)$ level the concept of pole mass is well defined. The uncertainty of the measured $t$-quark mass is about $650 - 750$ MeV:
$M_t^{exp} \approx 174.30 \pm 0.35(stat) \pm 0.54(syst)$ GeV *(Tevatron, (l+jets)-channel, 2018)*;
$M_t^{exp} \approx 173.34 \pm 0.27(stat) \pm 0.71(syst)$ GeV (*combination of results ATLAS, CMS, 2014*).

# $\overline{\text{MS}}$-on-shell relation in QED

Using the $U(1)$-limit of the QCD results with $SU(N_c)$ gauge group we obtain that the $\mathcal{O}(a^4)$ contribution to the $z_m$-ratio can be represented as:

$$z^{(4)}_{m,\,\text{QED}}(M_l^2) = 4.06885 n_l^3 - 2.3576 n_l^2 + (-4.097 \pm 0.178) n_l - 10.761 \pm 1.030 \ .$$

Thus, for $e, \mu$ and $\tau$-leptons the following expansions hold at $\mu^2 = M_l^2$:

$$\begin{aligned}
M_e &\approx \overline{m}_e(M_e^2)(1 + a + 1.66591 a^2 - 2.02839 a^3 + (5.482 \pm 1.030) a^4) \ , \\
M_\mu &\approx \overline{m}_\mu(M_\mu^2)(1 + a + 0.10386 a^2 - 3.96938 a^3 + (5.907 \pm 1.045) a^4) \ , \\
M_\tau &\approx \overline{m}_\tau(M_\tau^2)(1 + a - 1.45819 a^2 - 1.99421 a^3 + \underline{(-0.653 \pm 1.090)} a^4) \ .
\end{aligned}$$

or at $\mu^2 = \overline{m}_q^2$:

$$\begin{aligned}
M_e &\approx \overline{m}_e(\overline{m}_e^2)(1 + \overline{a} + 0.16591 \overline{a}^2 - 2.13144 \overline{a}^3 + (7.487 \pm 1.030) \overline{a}^4) \ , \\
M_\mu &\approx \overline{m}_\mu(\overline{m}_\mu^2)(1 + \overline{a} - 1.39614 \overline{a}^2 - 0.64601 \overline{a}^3 + (3.169 \pm 1.045) \overline{a}^4) \ , \\
M_\tau &\approx \overline{m}_\tau(\overline{m}_\tau^2)(1 + \overline{a} - 2.95819 \overline{a}^2 + 4.75557 \overline{a}^3 + \underline{(-21.238 \pm 1.090)} \overline{a}^4) \ .
\end{aligned}$$

The presented formulas demonstrate the absence of any sign-constant or sign-alternating structure of series of PT in QED for $\overline{\text{MS}}$-on-shell relation.

# Estimates of the multiloop corrections by the ECH-motivated method

The effective charges (ECH)-motivated method *(Kataev, Starshenko, 95)* gives possibility to estimate high-order corrections to the mass conversion formula *(Kataev, Kim, 2010)*. We start from the Euclidean region with $\mu^2 = Q^2$ and take into account effects of the analytical continuation to the Minkowskian space with $\mu^2 = s$.
As the associated RG function, determined in the Euclidean region, we put $F(Q^2)$-function, related to its image $T(s)$ in the Minkowskian space through the Källen-Lehmann type spectral representation *(Chetyrkin, Kniehl, Sirlin, 1997)*:

$$F(Q^2) = Q^2 \int\limits_0^\infty ds \frac{T(s)}{(s+Q^2)^2} \ ,$$

$$T(s) = \overline{m}_q(s) \sum_{n=0}^{\infty} t_n^M a_s^n(s) \ , \qquad F(Q^2) = \overline{m}_q(Q^2) \sum_{n=0}^{\infty} f_n^E a_s^n(Q^2) \ .$$

# $\pi^2$-effects

The integration gives:

$$Q^2 \int_0^\infty ds \frac{\{1; l; l^2; l^3; l^4; l^5; l^6\}}{(s+Q^2)^2} = \Big\{1;\ \mathfrak{L};\ \mathfrak{L}^2 + \frac{\pi^2}{3};\ \mathfrak{L}^3 + \pi^2\mathfrak{L};\ \mathfrak{L}^4 + 2\pi^2\mathfrak{L}^2 + \frac{7\pi^4}{15};$$

$$\mathfrak{L}^5 + \frac{10}{3}\pi^2\mathfrak{L}^3 + \frac{7}{3}\pi^4\mathfrak{L};\ \mathfrak{L}^6 + 5\pi^2\mathfrak{L}^4 + 7\pi^4\mathfrak{L}^2 + \frac{31}{21}\pi^6\Big\}$$

with $l = \log(\mu^2/s)$ and $\mathfrak{L} = \log(\mu^2/Q^2)$.
Fixing $\mu^2 = Q^2$ we obtain the relation between the above mentioned coefficients $t_n^M$ and $f_n^E$ with given from integration $\pi^2$-effects. This relation can be written as $f_n^E = t_n^M + \Delta_n$ and is presented as:

$$\Delta_0 = 0\,, \qquad \Delta_1 = 0\,, \qquad \Delta_2 = \frac{\pi^2}{6}\gamma_0(\beta_0+\gamma_0)t_0^M\,,$$

$$\Delta_3 = \frac{\pi^2}{3}\left[t_1^M(\beta_0+\gamma_0)\left(\beta_0+\frac{1}{2}\gamma_0\right) + t_0^M\left(\frac{1}{2}\beta_1\gamma_0 + \gamma_1\beta_0 + \gamma_1\gamma_0\right)\right]\,,$$

$$\Delta_4 = \frac{\pi^2}{3}\left[t_2^M\left(3\beta_0^2 + \frac{5}{2}\beta_0\gamma_0 + \frac{1}{2}\gamma_0^2\right) + t_1^M\left(\frac{3}{2}\beta_1\gamma_0 + \frac{5}{2}\beta_1\beta_0 + 2\gamma_1\beta_0 + \gamma_1\gamma_0\right)\right.$$

$$\left. + \quad t_0^M\left(\frac{1}{2}\beta_2\gamma_0 + \gamma_1\beta_1 + \frac{1}{2}\gamma_1^2 + \frac{3}{2}\gamma_2\beta_0 + \gamma_2\gamma_0\right)\right] + \frac{7\pi^4}{60}t_0^M\gamma_0(\beta_0+\gamma_0)\left(\beta_0+\frac{1}{2}\gamma_0\right)$$

# $\pi^2$-effects

$$
\begin{aligned}
\Delta_5 \quad = \quad & \frac{\pi^2}{3}\Bigg[t_3^M\left(6\beta_0^2+\frac{7}{2}\beta_0\gamma_0+\frac{1}{2}\gamma_0^2\right)+t_2^M\left(7\beta_1\beta_0+3\gamma_1\beta_0+\frac{5}{2}\beta_1\gamma_0+\gamma_1\gamma_0\right) \\
+ \quad & t_1^M\left(\frac{3}{2}\beta_1^2+\frac{1}{2}\gamma_1^2+3\beta_2\beta_0+\frac{5}{2}\gamma_2\beta_0+2\beta_1\gamma_1+\frac{3}{2}\beta_2\gamma_0+\gamma_2\gamma_0\right) \\
+ \quad & t_0^M\left(\frac{1}{2}\beta_3\gamma_0+\beta_2\gamma_1+\frac{3}{2}\gamma_2\beta_1+2\gamma_3\beta_0+\gamma_1\gamma_2+\gamma_0\gamma_3\right)\Bigg] \\
+ \quad & \frac{7\pi^4}{15}\Bigg[t_1^M\left(\beta_0^4+\frac{25}{12}\beta_0^3\gamma_0+\frac{35}{24}\beta_0^2\gamma_0^2+\frac{5}{12}\beta_0\gamma_0^3+\frac{1}{24}\gamma_0^4\right) \\
+ \quad & t_0^M\left(\gamma_1\beta_0^3+\frac{13}{12}\gamma_0\beta_1\beta_0^2+\frac{13}{12}\gamma_0^2\beta_0\beta_1+\frac{11}{6}\gamma_0\gamma_1\beta_0^2+\gamma_0^2\beta_0\gamma_1+\frac{1}{4}\beta_1\gamma_0^3+\frac{1}{6}\gamma_1\gamma_0^3\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\Delta_6 &= \frac{\pi^2}{3}\Bigg[t_4^M\left(10\beta_0^2+\frac{9}{2}\beta_0\gamma_0+\frac{1}{2}\gamma_0^2\right)+t_3^M\left(\frac{27}{2}\beta_0\beta_1+4\beta_0\gamma_1+\frac{7}{2}\beta_1\gamma_0+\gamma_0\gamma_1\right)\\
&+ t_2^M\left(8\beta_0\beta_2+\frac{7}{2}\beta_0\gamma_2+3\beta_1\gamma_1+\frac{5}{2}\beta_2\gamma_0+4\beta_1^2+\frac{1}{2}\gamma_1^2+\gamma_0\gamma_2\right)\\
&+ t_1^M\left(\frac{7}{2}\beta_0\beta_3+\frac{7}{2}\beta_1\beta_2+3\beta_0\gamma_3+\frac{5}{2}\beta_1\gamma_2+2\beta_2\gamma_1+\frac{3}{2}\beta_3\gamma_0+\gamma_0\gamma_3+\gamma_1\gamma_2\right)\\
&+ t_0^M\left(\frac{1}{2}\gamma_2^2+\frac{3}{2}\beta_2\gamma_2+\frac{5}{2}\beta_0\gamma_4+2\beta_1\gamma_3+\beta_3\gamma_1+\frac{1}{2}\beta_4\gamma_0+\gamma_0\gamma_4+\gamma_1\gamma_3\right)\Bigg]\\
&+ \frac{7\pi^4}{15}\Bigg[t_2^M\left(5\beta_0^4+\frac{77}{12}\beta_0^3\gamma_0+\frac{71}{24}\beta_0^2\gamma_0^2+\frac{7}{12}\beta_0\gamma_0^3+\frac{1}{24}\gamma_0^4\right)+t_1^M\Bigg(\frac{77}{12}\beta_0^3\beta_1\\
&+ \frac{5}{12}\beta_1\gamma_0^3+4\beta_0^3\gamma_1+\frac{1}{6}\gamma_0^3\gamma_1+\frac{10}{3}\beta_0\beta_1\gamma_0^2+\frac{25}{3}\beta_0^2\beta_1\gamma_0+\frac{3}{2}\beta_0\gamma_0^2\gamma_1+\frac{13}{3}\beta_0^2\gamma_0\gamma_1\Bigg)\\
&+ t_0^M\Bigg(\frac{1}{4}\beta_2\gamma_0^3+\frac{5}{2}\beta_0^3\gamma_2+\frac{1}{6}\gamma_0^3\gamma_2+\frac{3}{2}\beta_0^2\gamma_1^2+\frac{5}{8}\beta_1^2\gamma_0^2+\frac{1}{4}\gamma_0^2\gamma_1^2+\frac{35}{24}\beta_0\beta_1^2\gamma_0\\
&+ \frac{5}{4}\beta_0\beta_2\gamma_0^2+\frac{47}{12}\beta_0^2\beta_1\gamma_1+\frac{3}{2}\beta_0^2\beta_2\gamma_0+\frac{5}{4}\beta_0\gamma_0\gamma_1^2+\frac{5}{4}\beta_0\gamma_0^2\gamma_2+\beta_1\gamma_0^2\gamma_1\\
&+ \frac{37}{12}\beta_0^2\gamma_0\gamma_2+\frac{25}{6}\beta_0\beta_1\gamma_0\gamma_1\Bigg)\Bigg]\\
&+ \frac{31\pi^6}{126}t_0^M\gamma_0(\beta_0+\gamma_0)\left(\beta_0+\frac{1}{2}\gamma_0\right)\left(\beta_0+\frac{1}{3}\gamma_0\right)\left(\beta_0+\frac{1}{4}\gamma_0\right)\left(\beta_0+\frac{1}{5}\gamma_0\right).
\end{aligned}
$$

# ECH-motivated approach

For $SU_c(3)$ case we have:

$$\begin{aligned}
\Delta_2 &= 5.89434 - 0.274156 n_l \ , \\
\Delta_3 &= 105.6221 - 10.04477 n_l + 0.198002 n_l^2 \ , \\
\Delta_4 &= 2272.002 - 403.9489 n_l + 20.67673 n_l^2 - 0.315898 n_l^3 \ , \\
\Delta_5 &= 56304.639 - 13767.2725 n_l + 1137.17794 n_l^2 - 37.745285 n_l^3 + 0.427523 n_l^4 \ , \\
\Delta_6 &= 1633115.62 \pm 347.65 + (-518511.694 \pm 56.723) n_l + (61128.1666 \pm 4.7791) n_l^2 \\
&+ (-3345.0818 \pm 0.1371) n_l^3 + 85.37937 n_l^4 - 0.818446 n_l^5 .
\end{aligned}$$

The next stage is to determine the effective charge $a_s^{eff}(Q^2)$ for Euclidean quantity $F(Q^2)/\overline{m}_q(Q^2)$:

$$\frac{F(Q^2)}{\overline{m}_q(Q^2)} = f_0^E + f_1^E a_s^{eff}(Q^2) \ , \qquad a_s^{eff}(Q^2) = a_s(Q^2) + \sum_{k=2}^{\infty} \phi_k a_s^k(Q^2) \ ,$$

where terms $\phi_k$ are equal to $\phi_k = f_k^E / f_1^E$.

# ECH-motivated approach

After this we can define the ECH $\beta$-function for $a_s^{eff}(Q^2)$:

$$
\begin{aligned}
\beta_0^{eff} &= \beta_0\ , \qquad \beta_1^{eff} = \beta_1\ , \qquad \beta_2^{eff} = \beta_2 - \phi_2\beta_1 + (\phi_3 - \phi_2^2)\beta_0\ , \\
\beta_3^{eff} &= \beta_3 - 2\phi_2\beta_2 + \phi_2^2\beta_1 + (2\phi_4 - 6\phi_2\phi_3 + 4\phi_2^3)\beta_0\ , \\
\beta_4^{eff} &= \beta_4 - 3\phi_2\beta_3 + (4\phi_2^2 - \phi_3)\beta_2 + (\phi_4 - 2\phi_2\phi_3)\beta_1 \\
&+ (3\phi_5 - 12\phi_2\phi_4 - 5\phi_3^2 + 28\phi_2^2\phi_3 - 14\phi_2^4)\beta_0\ , \\
\beta_5^{eff} &= \beta_5 - 4\phi_2\beta_4 + (8\phi_2^2 - 2\phi_3)\beta_3 + (4\phi_2\phi_3 - 8\phi_2^3)\beta_2 \\
&+ (2\phi_5 - 8\phi_2\phi_4 + 16\phi_2^2\phi_3 - 3\phi_3^2 - 6\phi_2^4)\beta_1 \\
&+ (4\phi_6 - 20\phi_2\phi_5 - 16\phi_3\phi_4 + 48\phi_2\phi_3^2 - 120\phi_2^3\phi_3 \\
&+ 56\phi_2^2\phi_4 + 48\phi_2^5)\beta_0\ .
\end{aligned}
$$

The concrete form of the terms $t_n^M$ was not specified by us. We introduce the following expansion:

$$M_q = \overline{m}_q(\overline{m}_q^2)\sum_{n=0}^{\infty} t_n^M a_s^n(\overline{m}_q^2)\ .$$

# The essence of evaluation

If we would put $\beta_2^{eff} \approx \beta_2$, then we would get that $f_3^E \approx (f_2^E)^2/f_1^E + f_2^E \beta_1/\beta_0$ and using the relation $f_3^E = t_3^M + \Delta_3$ we would restore the value of $t_3^M$-term. Similarly, supposing that $\beta_3^{eff} \approx \beta_3$ we could estimate the value of the four-loop contribution $t_4^M$:

| $n_l$ | $t_3^{M,\, exact}$ | $t_3^{M,\, ECH}$ | $t_4^{M,\, exact}$ | $t_4^{M,\, ECH}$ |
|---|---|---|---|---|
| 3 | 116.494 | 124.097 | $1702.70 \pm 1.41$ | 1281.09 |
| 4 | 94.418 | 97.728 | $1235.66 \pm 1.47$ | 986.13 |
| 5 | 73.637 | 73.615 | $839.14 \pm 1.54$ | 719.38 |
| 6 | 54.161 | 51.775 | $509.07 \pm 1.61$ | 483.02 |
| 7 | 35.991 | 32.235 | $241.37 \pm 1.70$ | 279.37 |
| 8 | 19.126 | 15.034 | $31.99 \pm 1.80$ | 110.71 |

# The essence of evaluation

Therefore, we have reason to believe that conditions $\beta_4^{eff} \approx \beta_4$ and $\beta_5^{eff} \approx \beta_5$, and $f_5^E = t_5^M + \Delta_5$, $f_6^E = t_6^M + \Delta_6$ allow us to estimate values of $t_5^M$ and $t_6^M$-terms with satisfactory accuracy.

$$
\begin{aligned}
f_5^E &\approx \frac{1}{3\beta_0}\left[3f_2^E\beta_3 + \left(f_3^E - 4\frac{(f_2^E)^2}{f_1^E}\right)\beta_2 + \left(2\frac{f_2^E f_3^E}{f_1^E} - f_4^E\right)\beta_1\right] \\
&+ 4\frac{f_2^E f_4^E}{f_1^E} + \frac{5}{3}\frac{(f_3^E)^2}{f_1^E} - \frac{28}{3}f_3^E\left(\frac{f_2^E}{f_1^E}\right)^2 + \frac{14}{3}\frac{(f_2^E)^4}{(f_1^E)^3}\,, \\
f_6^E &\approx \frac{1}{4\beta_0}\left[4f_2^E\beta_4 + \left(2f_3^E - 8\frac{(f_2^E)^2}{f_1^E}\right)\beta_3 + \left(8\frac{(f_2^E)^3}{(f_1^E)^2} - 4\frac{f_2^E f_3^E}{f_1^E}\right)\beta_2\right. \\
&+ \left.\left(6\frac{(f_2^E)^4}{(f_1^E)^3} + 3\frac{(f_3^E)^2}{f_1^E} + 8\frac{f_2^E f_4^E}{f_1^E} - 16f_3^E\left(\frac{f_2^E}{f_1^E}\right)^2 - 2f_5^E\right)\beta_1\right] \\
&+ 5\frac{f_2^E f_5^E}{f_1^E} + 4\frac{f_3^E f_4^E}{f_1^E} + 30f_3^E\left(\frac{f_2^E}{f_1^E}\right)^3 - 12f_2^E\left(\frac{f_3^E}{f_1^E}\right)^2 - 12\frac{(f_2^E)^5}{(f_1^E)^4} - 14f_4^E\left(\frac{f_2^E}{f_1^E}\right)^2.
\end{aligned}
$$

# Numerical results

| $n_l$ | $t_5^{M,\,ECH}$ | $t_6^{M,\,ECH}$ |
|---|---|---|
| 3 | 28435 | 476522 |
| 4 | 17255 | 238025 |
| 5 | 9122 | 90739 |
| 6 | 3490 | 8412 |
| 7 | -127 | -29701 |
| 8 | -2153 | -39432 |

# Numerical results

Taking into account that the five-loop contribution $t_5^M$ can be expanded in powers of number of massless flavors in the form $t_5^M = t_{54}^M n_l^4 + t_{53}^M n_l^3 + t_{52}^M n_l^2 + t_{51}^M n_l + t_{50}^M$ with unknown variables $t_{54}^M - t_{50}^M$, we obtain the following matrix equation for number of $n_l$, equal to number of these unknown variables, namely for $3 \le n_l \le 7$:

$$\begin{pmatrix} 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \\ 1 & 6 & 36 & 216 & 1296 \\ 1 & 7 & 49 & 343 & 2401 \end{pmatrix} \begin{pmatrix} t_{50}^{M,\,ECH} \\ t_{51}^{M,\,ECH} \\ t_{52}^{M,\,ECH} \\ t_{53}^{M,\,ECH} \\ t_{54}^{M,\,ECH} \end{pmatrix} = \begin{pmatrix} 28435 \\ 17255 \\ 9122 \\ 3490 \\ -127 \end{pmatrix}$$

The numerical solution of this system with the Vandermonde matrix can be written as

$$t_5^{M,\,ECH} = 2.5 n_l^4 - 136 n_l^3 + 2912 n_l^2 - 26976 n_l + 86620 \ .$$

Repeating the similar reasoning for $t_6^M$-contribution with $3 \le n_l \le 8$, we obtain

$$t_6^{M,\,ECH} = -4.9 n_l^5 + 352 n_l^4 - 9708 n_l^3 + 131176 n_l^2 - 855342 n_l + 2096737 \ .$$

# Numerical results

Thus, we arrive to the following expansions within the framework of the ECH approach:

$$\frac{M_c}{1~\text{GeV}} \approx 1.28 + 0.207 + 0.195 + 0.268 + 0.475 + \boxed{0.965 + 1.965}\,,$$
$$\frac{M_b}{1~\text{GeV}} \approx 4.18 + 0.399 + 0.199 + 0.145 + \underline{0.136 + \boxed{0.136 + 0.136}}\,,$$
$$\frac{M_t}{1~\text{GeV}} \approx 163.5 + 7.543 + 1.612 + 0.499 + 0.197 + \boxed{0.074 + 0.025}\,.$$

The boxed terms are estimated using the method of effective charges. Despite the fact that these formulas are approximate, they reflect the specific behavior of the $\overline{\text{MS}}$-on-shell relation in the higher orders of PT. For $b$-quark the ECH approach demonstrates a rather cunning behavior of the PT series for its pole mass. We observe some kind of the island of stability. The four, five and six-loop contributions coincide literally. The series for $t$-quark shows a decrease of the $\mathcal{O}(a_s^5)$ and $\mathcal{O}(a_s^6)$-contributions. Thus, with a high degree of probability the concept of pole mass of $t$-quark can be used even at the six-loop level. Therefore we can sum all these corrections and we obtain $M_t^{ECH} \approx 173.45$ GeV.

# Comparison with the renormalon-based analysis

The renormalon dominance hypothesis leads to the following factorial growth of the $t_n^M$-corrections at $\mu^2 = \overline{m}_q^2$ renormalization point *(Beneke, Braun, 94-95)* ,

$$t_n^{M,\, r-n} \xrightarrow{n\to\infty} \pi N_m (2\beta_0)^{n-1} \frac{\Gamma(n+b)}{\Gamma(1+b)} \Bigg(1 + \frac{s_1}{n+b-1} + \frac{s_2}{(n+b-1)(n+b-2)} + \frac{s_3}{(n+b-1)(n+b-2)(n+b-3)} + \mathcal{O}\left(\frac{1}{n^4}\right)\Bigg) ,$$

where $\Gamma(x)$ is the Euler Gamma-function, $b = \beta_1/(2\beta_0^2)$. The normalization factor $N_m$ depends on $n_l$ and on the order of PT and can not be obtained rigorously by PT.

$$\begin{aligned}
s_1 &= \frac{1}{4\beta_0^4}(\beta_1^2 - \beta_0\beta_2) , \\
s_2 &= \frac{1}{32\beta_0^8}(\beta_1^4 - 2\beta_1^3\beta_0^2 - 2\beta_1^2\beta_2\beta_0 + 4\beta_1\beta_2\beta_0^3 + \beta_2^2\beta_0^2 - 2\beta_3\beta_0^4) , \\
s_3 &= \frac{1}{384\beta_0^{12}}(\beta_1^6 - 6\beta_1^5\beta_0^2 + 8\beta_1^4\beta_0^4 - 3\beta_1^4\beta_2\beta_0 + 18\beta_1^3\beta_2\beta_0^3 - 24\beta_1^2\beta_2\beta_0^5 \\
&+ 3\beta_1^2\beta_2^2\beta_0^2 - 6\beta_1^2\beta_3\beta_0^4 - 12\beta_1\beta_2^2\beta_0^4 + 16\beta_1\beta_3\beta_0^6 - \beta_2^3\beta_0^3 + 8\beta_2^2\beta_0^6 \\
&+ 6\beta_2\beta_3\beta_0^5 - 8\beta_4\beta_0^7)
\end{aligned}$$

*(Beneke, Marquard, Nason, Steinhauser, 2017)* .

# Comparison with the renormalon-based analysis

Based on the results *(Pineda, 2001)*, *(Beneke, Marquard, ... 2017)* we propose to put $N_m \approx 0.5$ for charm, bottom and top-quark in five and six-loop approximation. Thus, we find:

| $n_l$ | $t_5^{M,\, r-n}$ | $t_6^{M,\, r-n}$ |
|---|---|---|
| 3 | 31527 | 768520 |
| 4 | 22335 | 501230 |
| 5 | 15089 | 308590 |

The obtained values of the five and six-loop corrections outline the following behavior of the PT series for pole masses of heavy quarks with renormalon asymptotic:

$$\frac{M_c}{1\ \text{GeV}} \approx 1.28 + 0.207 + 0.195 + 0.268 + 0.475 + \boxed{1.070 + 3.170}\ ,$$

$$\frac{M_b}{1\ \text{GeV}} \approx 4.18 + 0.399 + 0.199 + 0.145 + 0.136 + \boxed{0.177 + 0.284}\ ,$$

$$\frac{M_t}{1\ \text{GeV}} \approx 163.5 + 7.543 + 1.612 + 0.499 + 0.197 + \boxed{0.122 + 0.087}\ .$$

# Comparison with the renormalon-based analysis

Renormalon dominance hypothesis with $N_m \approx 0.5$ allows to obtain the following evaluations:

$$\frac{M_t}{1\ \text{GeV}} \approx 163.5 + 7.543 + 1.612 + 0.499 + 0.197 + \boxed{0.122 + 0.087}$$
$$+ \boxed{0.073 + 0.071 + 0.078 + 0.097 + \ldots}$$

This estimate procedure permit us to understand approximately, from what level of PT the asymptotic behavior of the QCD series for pole mass of $t$-quark begins to manifest itself. The first traces of this effect can already be observed in the seven order of PT. The eighth and ninth contributions are either comparable or exceed the value of the seventh correction.

# Comparison of the two considered methods

$$\frac{M_c}{1\ \text{GeV}} \overset{\text{ECH}}{\approx} 1.28 + 0.207 + 0.195 + 0.268 + 0.475 + \boxed{0.965 + 1.965}\,,$$

$$\frac{M_c}{1\ \text{GeV}} \overset{\text{r-n}}{\approx} 1.28 + 0.207 + 0.195 + 0.268 + 0.475 + \boxed{1.070 + 3.170}\,,$$

$$\frac{M_b}{1\ \text{GeV}} \overset{\text{ECH}}{\approx} 4.18 + 0.399 + 0.199 + 0.145 + 0.136 + \boxed{0.136 + 0.136}\,,$$

$$\frac{M_b}{1\ \text{GeV}} \overset{\text{r-n}}{\approx} 4.18 + 0.399 + 0.199 + 0.145 + 0.136 + \boxed{0.177 + 0.284}\,,$$

$$\frac{M_t}{1\ \text{GeV}} \overset{\text{ECH}}{\approx} 163.5 + 7.543 + 1.612 + 0.499 + 0.197 + \boxed{0.074 + 0.025}\,,$$

$$\frac{M_t}{1\ \text{GeV}} \overset{\text{r-n}}{\approx} 163.5 + 7.543 + 1.612 + 0.499 + 0.197 + \boxed{0.122 + 0.087}$$

$$+ \boxed{0.073 + 0.071 + 0.078 + 0.097}\,.$$

# Conclusion

- We evaluate the two unknown in analytical form four-loop coefficients $z_m^{(40)}$ and $z_m^{(41)}$ and their uncertainties by the LSM in QCD and QED
- Applying the ECH-motivated approach with arising $\pi^2$-effects from the analytic continuation from the Euclidean to Minkowskian space we obtain five and six-loop contributions to the QCD $\overline{\text{MS}}$-on-shell relation.
- We indicate that ECH-motivared method for bottom-quark pole mass leads to the effect of a plateau, whereas for top-quark the five and six-loop corrections are decreased.
- In the framework of the renormalon-dominated hypothesis we estimate $\mathcal{O}(a_s^5)$ and $\mathcal{O}(a_s^6)$-contributions to the pole mass of $c$, $b$ and $t$-quarks. The results of this hypothesis show different behavior of these corrections for $b$-quark and similar for $t$-quark.
- The renormalon-based analysis is applied up to 10 order of PT and we conclude that the asymptotic behavior for expansion of the pole mass of top-quark through its running mass begins to manifest itself somewhere at the 7 or 8 level of PT. Therefore the concept of pole mass of top-quark can be safely considered in the phenomenology studies.

Thank you for your attention!