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SUSY-like relation in evolution of gluon and quark jet multiplicities

OUTLINE

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GENERAL

The cross-sections σ, which studied at high-energy colliders as LHC, can be represented (symbolically) as

$$\sigma = P \otimes \hat{\sigma} \otimes D, \tag{1}$$

with the Mellin convolution

$$A \otimes B \equiv \int_x^1 \frac{dy}{y} A(y) B\left(\frac{x}{y}\right) . \tag{2}$$

Here $\hat{\sigma}$ is perturbatively calculated ("hard") parton-parton cross-sections,
P are the parton distribution functions (PDFs) or parton luminocities ($=$ Mellin convolution of two PDFs),
D are the fragmentation fanctions (FFs).

P and D are universal [= process-independent!!!] functions but scheme-dependent ones (usually $\overline{MS}$ is used).
P [D] contains all information about the transformation hadrons to (free) partons [(free) partons to hadrons, i.e. **hadronization**].
So, P and D are strongly dependent on strong interactions (of quarks and gluons in hadrons) and, thus, they cannot be calculated in the framework of the perturbation theory. They (really, their normalizations) should be taken from experiment.

$P = P(x, \mu^2)$ and $D = D(x, \mu^2)$, where x is Bjorken variable (= ratio of parton momentum to hadron one in P
(and the corresponding ratio of hadron momentum to parton one in D) and, thus, $0 \leq x \leq 1$) and μ^2 is some additional variable (for example, in DIS $\mu^2 = Q^2$ is the “mass” of photon).

At $\mu^2 \to \infty$, $P(x,\mu^2) \to P(x)$ and $D(x,\mu^2) \to D(x)$, i.e. we have Feynman Parton Model.

The evolution of $D = D(x,\mu^2)$ [and $P = P(x,\mu^2)$] is perturbatively calculatable (DGLAP equations):

$$\mu^2 \frac{d}{d\mu^2} \hat{D}(x,\mu^2) = \hat{P}(x,\mu^2) \otimes \hat{D}(x,\mu^2),$$

$$\hat{D}(x,\mu^2) = \begin{pmatrix} D_q(x,\mu^2) \\ D_g(x,\mu^2) \end{pmatrix}, \quad \hat{P}(x,\mu^2) = \begin{pmatrix} P_{qq}(x,\mu^2) & P_{qg}(x,\mu^2) \\ P_{gq}(x,\mu^2) & P_{gg}(x,\mu^2) \end{pmatrix},$$

where $\hat{P}$ is the matrix of the perturbative kernels, which are known at 3-loop orders.

It is convenient to do the Mellin transform of above DGLAP matrix equation. In Mellin moment space the corresponding DGLAP equations become to be pure differential ones:

$$\mu^2 \frac{d}{d\mu^2} \hat{D}(N, \mu^2) = \hat{P}(N) \hat{D}(N, \mu^2), \quad A(N) = \int_0^1 dx\, x^{N-1} A(x)\,. \tag{3}$$

The important property is diagonalization of (3).

After diagonalization in both the parts there are two components: "$+$" one and "$-$" one.

The "+" component contains singularities at $N \to 1$ in the corresponding anomalous dimensions and coefficient functions (N is Mellin moment number) and requires some resummations. Usually the "+" component is the object of study **(MLLA approach, for example)**.

The "−" component is free of singularities at $N \to 1$ in the corresponding anomalous dimensions and coefficient functions. It has very slow Q^2-dependence (in both the considered cases).
!!! But it is strongly necessary for the agreement with experimental data. !!!

Really, namely the absence in the past analyses of the "−" component is resonable for a strong disagreement between theory and experiment.

1. Average Multiplicities

- I present the rather old results (B.Bolzoni, B.A. Kniehl and A.V.K., 2013) for gluon and quark average multiplicities, which are motivated by recent progress in timelike small-x resummation obtained in the $\overline{\mathrm{MS}}$ scheme. (C.-H.Korn, A. Vogt and K.Yeats, 2012).
  The results contain the next-to-next-to-leading-logarithmic (NNLL) resummed expressions and depend on two nonperturbative parameters with clear and simple physical interpretations.

- I present the new results (B.A. Kniehl and A.V.K., 2017) for gluon and quark average multiplicities.
  The results contain a new SUSY-like relations between the corresponding anomalous dimensions (i.e. the first moments od splitting functions).

- We did a global fit of these two quantities. Our results solved a longstanding problem of QCD: a disagreement between theoretical predictions for the ration of gluon and quark average multiplicities and the corresponding experimental data.

- We finally proposed also to use the multiplicity data as a new way to extract the strong-coupling constant. We obtained $\alpha_s^{(5)}(M_z) = 0.1205 \pm 0.0020$ in the $\overline{\mathrm{MS}}$ scheme in an approximation equivalent to next-to-next-to-leading order (NNNLO) enhanced by the resummations of $\ln x$ terms through the NNLL level, in excellent agreement with the present world average.

2. Introduction

The computation of average jet multiplicities (i.e. $D_a(N=1,\mu^2)$, $a=g,q$) requires small-x resummation, (A.H.Mueller, 1981). [**Similar resummation was done in** (R. Kirschner and L.N. Lipatov, 1982)].

It was shown that the singularities for $x \sim 0$, which are encoded in large logarithms of the kind $\ln^k(1/x)$

(the k-th term in the perturbative theory contains the terms like a_s^k/ω^{2k-1} in the Mellin space [hereafter $\omega = N-1$])

and disappear after resummation. Usually, the resummation includes the singularities from all orders according to a certain logarithmic accuracy, for which it *restores* perturbation theory.

Example, in the Mellin space, N is Mellin moment, $\omega = N - 1$:

$$a_s\left(\frac{1}{\omega} + Const\right) + a_s^2\left(\sim \frac{1}{\omega^3} + ...\right) + a_s^3\left(\sim \frac{1}{\omega^5} + ...\right) + ...$$
$$\to a_s\left(\frac{1}{\omega_{\rm eff}} + Const\right) \quad \text{LL-resummation},$$
$$\frac{a_s}{\omega} \to \frac{a_s}{\omega_{\rm eff}} = \frac{a_s}{\omega}\frac{1}{\sqrt{1+\frac{8C_A a_s}{\omega^2}}} = \frac{a_s}{\sqrt{\omega^2 + 8C_A a_s}} \to \sqrt{\frac{a_s}{8C_A}} \quad (\text{at} \;\; \omega \to 0)\,,$$

i.e.

$$\omega \to \omega_{\rm eff} = \sqrt{\omega^2 + 8C_A a_s} \to \sqrt{8C_A a_s} \quad (\text{at} \;\; \omega \to 0)\,.$$

!!!So, after resummation we have perturbation theory in the parameter $\sqrt{a_s}$, not a_s.!!!

There was also a strong disagreement between theory and experiment. Indeed,

$$r \equiv \frac{D_g(0,\mu^2)}{D_q(0,\mu^2)} = \frac{c_A}{c_F} \cdot (1 + O(a_s)) = 2.25 \cdot (1 + O(a_s)) \qquad (4)$$

is equal to 2.25 at $\mu^2 \to \infty$.

Experimental data show the results for r two times less.

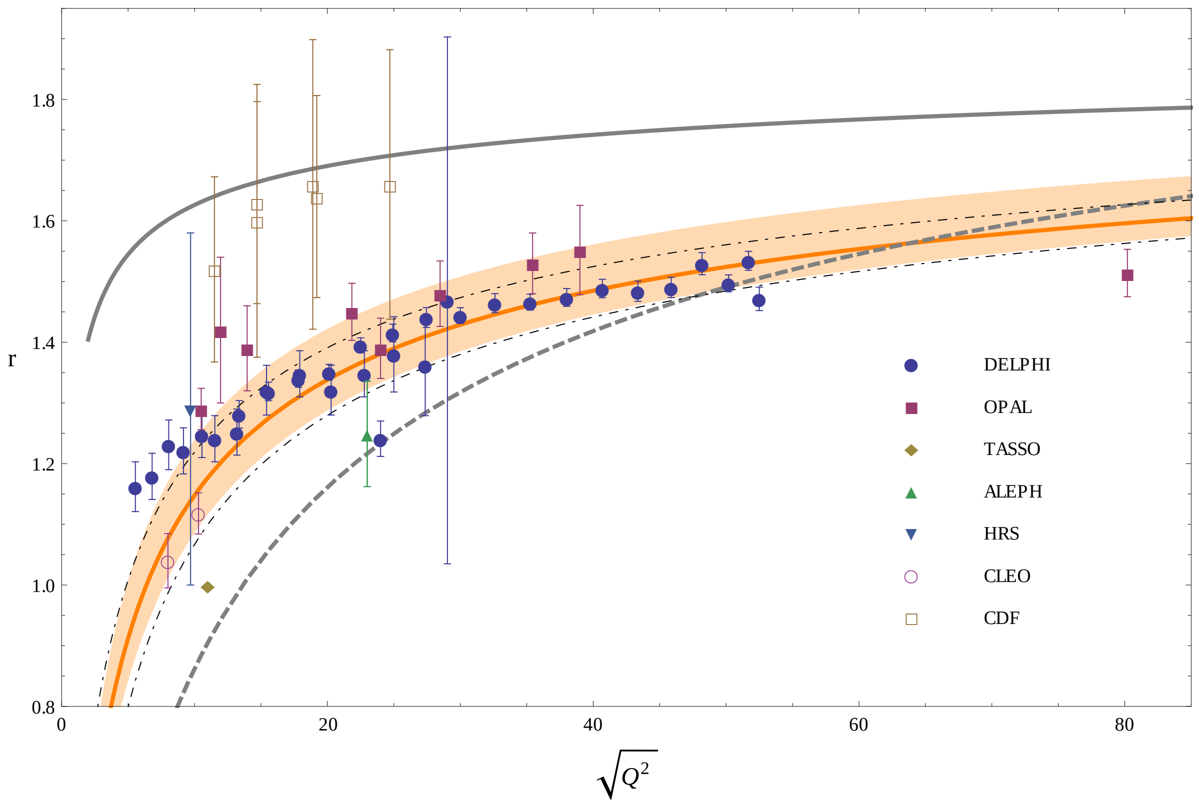

Figure 1: The average gluon-to-quark jet multiplicity ratio evaluated in the LO + NNLL (dashed/gray lines) and N^{3}L (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from Table **??** are con mental data. The experimental and theoretical uncertainties in the N^{3}LO$_{\text{approx+NLO+NNLL}}$ result are indicated by the shaded/or bands enclosed between the dot-dashed curves, respectively. The prediction given by analysis in (A.Capella, I.M.Dremin, J.W.G and J. Tran Thanh Van, 2000) is indicated by the continuous/gray line.

3. Fragmentation functions and their evolution

The evolution of the fragmentation functions $D_a(x,\mu^2)$ for the gluon–quark-singlet system $a = g, q$. In Mellin space, is:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} D_q(\omega,\mu^2) \\ D_g(\omega,\mu^2) \end{pmatrix} = \begin{pmatrix} P_{qq}(\omega,a_s) & P_{gq}(\omega,a_s) \\ P_{qg}(\omega,a_s) & P_{gg}(\omega,a_s) \end{pmatrix} \begin{pmatrix} D_q(\omega,\mu^2) \\ D_g(\omega,\mu^2) \end{pmatrix}, \quad (5)$$

where $P_{ij}(\omega, a_s)$, with $i, j = g, q$, are the timelike splitting functions, $\omega = N-1$, with N being the standard Mellin moments with respect to x, and $a_s(\mu^2) = \alpha_s(\mu)/(4\pi)$ is the couplant.

The standard definition of the hadron average multiplicities in terms of the fragmentation functions is given by their integral over x, which corresponds to the first Mellin moment, with $\omega = 0$:

$$\langle n_h(Q^2)\rangle_a \equiv \left[\int_0^1 dx\, x^\omega D_a(x,Q^2)\right]_{\omega=0} = D_a(\omega=0,Q^2) \qquad (6)$$

The timelike splitting functions $P_{ij}(\omega, a_s)$ may be computed perturbatively in a_s,

$$P_{ij}(\omega, a_s) = \sum_{k=0}^{\infty} a_s^{k+1} P_{ij}^{(k)}(\omega). \qquad (7)$$

The functions $P_{ij}^{(k)}(\omega)$ for $k = 0, 1, 2$ in the $\overline{\mathrm{MS}}$ scheme may be found through NNLO and with small-x resummation through NNLL accuracy.

4. Diagonalization

Standard approach (B.Bolzoni, B.A. Kniehl and A.V.K., 2013) (based on (A.J.Buras, 1980))

The *usual approach* is then to write a series expansion about the leading-order (LO) solution, which can in turn be diagonalized. One thus starts by choosing a basis in which the timelike-splitting-function matrix is diagonal at LO

$$P(\omega, a_s) = \begin{pmatrix} P_{++}(\omega, a_s) & P_{-+}(\omega, a_s) \\ P_{+-}(\omega, a_s) & P_{--}(\omega, a_s) \end{pmatrix}$$

$$= a_s \begin{pmatrix} P^{(0)}_{++}(\omega) & 0 \\ 0 & P^{(0)}_{--}(\omega) \end{pmatrix} + a_s^2 P^{(1)}(\omega) + O(a_s^3), \qquad (8)$$

with eigenvalues $P^{(0)}_{\pm\pm}(\omega)$.

It is convenient to represent the change of basis for the fragmentation functions order by order for $k \geq 0$:

$$\begin{aligned} D^+(\omega, \mu_0^2) &= (1-\alpha_\omega) D_s(\omega, \mu_0^2) - \epsilon_\omega D_g(\omega, \mu_0^2), \\ D^-(\omega, \mu_0^2) &= \alpha_\omega D_s(\omega, \mu_0^2) + \epsilon_\omega D_g(\omega, \mu_0^2). \end{aligned} \tag{9}$$

This implies for the components of the timelike-splitting-function matrix that

$$\begin{aligned} P_{--}^{(k)}(\omega) &= \alpha_\omega P_{qq}^{(k)}(\omega) + \epsilon_\omega P_{qg}^{(k)}(\omega) + \beta_\omega P_{gq}^{(k)}(\omega) + (1-\alpha_\omega) P_{gg}^{(k)}(\omega), \\ P_{-+}^{(k)}(\omega) &= P_{--}^{(k)}(\omega) - \left(P_{qq}^{(k)}(\omega) + \frac{1-\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right), \\ P_{++}^{(k)}(\omega) &= P_{qq}^{(k)}(\omega) + P_{gg}^{(k)}(\omega) - P_{--}^{(k)}(\omega), \\ P_{+-}^{(k)}(\omega) &= P_{++}^{(k)}(\omega) - \left(P_{qq}^{(k)}(\omega) - \frac{\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right) \\ &= P_{gg}^{(k)}(\omega) - \left(P_{--}^{(k)}(\omega) - \frac{\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right). \end{aligned} \tag{10}$$

The elements of the matrix for diagonalization **(LO projectors !!!)**

$$\alpha_\omega = \frac{P_{qq}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}, \quad \epsilon_\omega = \frac{P_{gq}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)},$$

$$\beta_\omega = \frac{P_{qg}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}. \qquad (11)$$

NEW approach (B.A. Kniehl and A.V.K., 2017)

Now we work directly with multiplicities, i.e. with the first Mellin moments $D_a(\mu^2) \equiv D_a(1,\mu^2)$ $a = q,g$ of the FFs $D_a(x,\mu^2)$, which in-turn obey the differential DGLAP equations in Mellin space:

($D_a(N,\mu^2) = \int_0^1 dx\, x^{N-1} D_a(x,\mu^2)$ **with** $N = 1, 2, \ldots$ **and similarly for** $P_{ba}(x)$**)**

with $P_{ba} = P_{ba}(N = 1, a_s(\mu^2))$

$$\frac{\mu^2 d}{d\mu^2}\begin{pmatrix} D_s(\mu^2) \\ D_g(\mu^2) \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{gq} \\ P_{qg} & P_{gg} \end{pmatrix} \begin{pmatrix} D_s(\mu^2) \\ D_g(\mu^2) \end{pmatrix}, \tag{12}$$

where $D_s = (1/2n_f)\sum_{q=1}^{n_f}(D_q + D_{\bar{q}})$, with n_f being the number of active quark flavors, is the quark singlet component.

The quark non-singlet component, which is irrelevant for the following, obeys a decoupled DGLAP equation.

So, our starting point is Eq. (12) with NNLL resummation **(with very special forms for P_{ab} with $a \neq b$)**

$$P_{aa} = \gamma_0(\delta_{ag} + K_a^{(1)}\gamma_0 + K_a^{(2)}\gamma_0^2) + O(\gamma_0^4), \quad (a = q, g),$$
$$P_{gq} = C(P_{gg} + A) + O(\gamma_0^4), \quad C = \frac{c_F}{c_A} = \frac{N_c^2 - 1}{2N_c^2} \ \text{for } SU(N_c) \text{ group}$$
$$P_{qg} = C^{-1}(P_{qq} + A) + O(\gamma_0^4), \tag{13}$$

where $\gamma_0 = \sqrt{2C_A a_s}$, with $a_s = \alpha_s/(4\pi)$ being the couplant, δ_{ab} is the Kronecker symbol, and

$$K_q^{(1)} = \frac{2}{3}C\varphi, \quad K_q^{(2)} = -\frac{1}{6}C\varphi[17 - 2\varphi(1 - 2C)],$$
$$K_g^{(1)} = -\frac{1}{12}[11 + 2\varphi(1 + 6C)], \quad K_g^{(2)} = \frac{1193}{288} - 2\zeta(2)$$
$$- \frac{5\varphi}{72}(7 - 38C) + \frac{\varphi^2}{72}(1 - 2C)(1 - 18C),$$
$$A = K_q^{(1)}\gamma_0^2, \quad \varphi = \frac{2n_f T_R}{C_A}, \quad T_R = \frac{1}{2}. \tag{14}$$

Eq. (13) is written in a form that allows us to glean a novel relationship:

$$C^{-1}P_{gq} - P_{gg} = CP_{qg} - P_{qq}, \tag{15}$$

which is independent of n_f.

Eq. (15) generalizes the case of SUSY QCD (SQCD)

$$P_{gq} - P_{gg} = P_{qg} - P_{qq}, \tag{16}$$

(Yu.L. Dokshitzer, 1977), (A.P. Bukhvostov, E.A. Kuraev, L.N. Lipatov, G.V. Frolov, 1985) from $C = 1$ to $C = 4/9$.

Eq.(16) exists for any N (or x) values but Eq.(15) is correct only at $N = 1$.

The relation (16) is known to be violated beyond LO in the standart dimensional regularization (DREG) scheme (but not in the dimensional reduction (DRED) scheme).

It will be interesting to see if Eq. (15) also holds beyond $O(\gamma_0^3)$, **et least in the case of the schemes, which preserve supersymmetry properties, such as the DRED.**

In above consideration (at $O(\gamma_0^3)$) the choice of a scheme is not so important because a difference in the results of various schemes is exactly canceled in Eq. (15).

We now solve Eq. (12) exactly by exploiting Eq. (15). To this end, we diagonalize the NNLL DGLAP evolution kernel as

$$U^{-1}\begin{pmatrix} P_{qq} & P_{gq} \\ P_{qg} & P_{gg} \end{pmatrix} U = \begin{pmatrix} P_{--} & 0 \\ 0 & P_{++} \end{pmatrix}, \tag{17}$$

by means of the matrices, **which are a_s-dependent now (i.e. Q^2-dependent)**

$$U = \begin{pmatrix} 1 & -1 \\ \frac{1-\alpha}{\varepsilon} & \frac{\alpha}{\varepsilon} \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} \alpha & \varepsilon \\ \alpha - 1 & \varepsilon \end{pmatrix}, \tag{18}$$

where

$$\alpha = \frac{P_{qq} - P_{++}}{P_{--} - P_{++}}, \qquad \varepsilon = \frac{P_{gq}}{P_{--} - P_{++}}, \tag{19}$$

$$P_{\pm\pm} = \frac{1}{2}\left[P_{qq} + P_{gg} \pm \sqrt{(P_{qq} - P_{gg})^2 + 4P_{qg}P_{gq}}\right]. \tag{20}$$

An indication of the new SUSY-like relation is the absence $\sqrt{\ldots}$ in the final results for $P_{\pm\pm}$, i.e. $(P_{qq} - P_{gg})^2 + 4P_{qg}P_{gq} = (\ldots)^2$

Owing to Eq. (15), the square root in Eq. (20) **is exactly canceled**, and we have **simple expressions for** $P_{\pm\pm}$

$$P_{--} = -A, \qquad P_{++} = P_{qq} + P_{gg} + A, \tag{21}$$

$$\alpha = \frac{P_{gg} + A}{P_{qq} + P_{gg} + 2A}, \qquad \varepsilon = -C\alpha\,. \tag{22}$$

!!! No $\sqrt{\ldots}$ in the final results (21) for $P_{\pm\pm}$

Eq. (12) thus assumes the form

$$\frac{\mu^2 d}{d\mu^2}\begin{pmatrix} D_- \\ D_+ \end{pmatrix} = \left[\begin{pmatrix} P_{--} & 0 \\ 0 & P_{++} \end{pmatrix} - U^{-1}\frac{\mu^2 d}{d\mu^2}U\right]\begin{pmatrix} D_- \\ D_+ \end{pmatrix}, \tag{23}$$

where the second term contained within the square brackets stems from the commutator of $\mu^2 d/d\mu^2$ and U, and

$$\begin{pmatrix} D_- \\ D_+ \end{pmatrix} = U^{-1}\begin{pmatrix} D_s \\ D_g \end{pmatrix} = \begin{pmatrix} \alpha D_s + \varepsilon D_g \\ (\alpha - 1)D_s + \varepsilon D_g \end{pmatrix}. \tag{24}$$

After some little algebra we may cast Eq. (12) in its final form,

$$\frac{\mu^2 d}{d\mu^2}\begin{pmatrix} D_- \\ D_+ \end{pmatrix} = \begin{pmatrix} \frac{C\varphi\beta_0}{3C_A}\gamma_0^3 - A & 0 \\ \frac{C\varphi\beta_0}{3C_A}\gamma_0^3 & P_{gg} + P_{qq} + A \end{pmatrix}\begin{pmatrix} D_- \\ D_+ \end{pmatrix}. \qquad (25)$$

The initial conditions are given by Eq. (24) for $\mu = \mu_0$ in terms of the three constants $\alpha_s(\mu_0^2)$, $D_s(\mu_0^2)$, and $D_g(\mu_0^2)$.

The solution of Eq. (25) is greatly facilitated by the fact that one entry of the matrix on its right-hand side is zero.

We may thus obtain D_- as the general solution of a homogeneous differential equation,

$$\frac{D_-(\mu^2)}{D_-(\mu_0^2)} = \exp\left[\int_{\mu_0^2}^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left(\frac{C\varphi\beta_0}{3C_A}\gamma_0^3 - A\right)\right] = \frac{T_-(\gamma_0(\mu^2))}{T_-(\gamma_0(\mu_0^2))}, \qquad (26)$$

where

$$\begin{aligned} T_-(\gamma_0) &= \exp\left[\frac{4C\varphi}{3}\int d\gamma_0\left(\frac{2C_A}{\beta_0\gamma_0} - 1\right)\right] \\ &= \gamma_0^{d_-}\exp\left(-\frac{4}{3}C\varphi\gamma_0\right), \end{aligned} \qquad (27)$$

with $d_- = 8C_AC\varphi/(3\beta_0)$. The small-$x$ correction $\propto \gamma_0$ in Eq. (27) originates from the extra term in Eq. (23) and represents a novel feature of our approach.

$D_-(\mu^2)$ has slow μ^2-dependence.

We are then left with an inhomogeneous differential equation for D_+. The general solution $\tilde{D}_+$ of its homogeneous part reads

$$\frac{\tilde{D}_+(\mu^2)}{\tilde{D}_+(\mu_0^2)} = \exp\left[\int_{\mu_0^2}^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\gamma_0 \left(1 + K_+^{(1)}\gamma_0 + K_+^{(2)}\gamma_0^2\right)\right] = \frac{T_+(\gamma_0(\mu^2))}{T_+(\gamma_0(\mu_0^2))},$$

where

$$\begin{aligned}
K_+^{(1)} &= 2K_q^{(1)} + K_g^{(1)} = -\frac{1}{12}[11 + 2\varphi(1-2C)], \\
K_+^{(2)} &= K_q^{(2)} + K_g^{(2)} = \frac{1193}{288} - 2\zeta(2) - \frac{7\varphi}{72}(5+2C) \\
&\quad + \frac{\varphi^2}{72}(1-2C)(1+6C), \\
T_+(\gamma_0) &= \gamma_0^{d_+} \exp\left[\frac{4C_A}{\beta_0\gamma_0} - \frac{4C_A}{\beta_0}\left(K_+^{(2)} - b_1\right)\gamma_0\right],
\end{aligned} \qquad (28)$$

with $d_+ = -4C_A K_+^{(1)}/\beta_0$ and $b_1 = \beta_1/(2C_A\beta_0)$.

Adding to $\tilde{D}_+$ a special solution of the inhomogeneous differential equation for D_+, we find its general solution to be

$$D_+(\mu^2) = \left[\frac{D_+(\mu_0^2)}{T_+(\gamma_0(\mu_0^2))} - \frac{4}{3}C\varphi\frac{D_-(\mu_0^2)}{T_-(\gamma_0(\mu_0^2))} \times \int_{\gamma_0(\mu_0^2)}^{\gamma_0(\mu^2)} \frac{d\gamma_0}{1+b_1\gamma_0^2}\frac{T_-(\gamma_0)}{T_+(\gamma_0)}\right] T_+(\gamma_0(\mu^2)). \quad (29)$$

The final expressions for D_- and D_+ in Eqs. (26) and (29), respectively, are fully renormalization group improved because all μ dependence resides in γ_0.

$D_+(\mu^2)$ has strong μ^2-dependence.

Using Eqs. (18) and (24), we now return to the parton basis, where it is useful to decompose $D_a = D_a^+ + D_a^-$ into the large and small components $D_a^\pm$ proportional to $D_\pm$, respectively. Defining $r_\pm = D_g^\pm / D_s^\pm$ and using Eqs. (13), (14), and (22), we then have $D_s^\pm = \mp D_\pm$ and

$$
\begin{aligned}
r_+ &= -\frac{\alpha}{\epsilon} = \frac{1}{C} + O(\gamma_0^2), \qquad (30)\\
r_- &= \frac{1-\alpha}{\epsilon} = -\frac{4}{3}\varphi\gamma_0 + \frac{\varphi}{18}[29 - 2\varphi(5-2C)]\gamma_0^2 + O(\gamma_0^3).
\end{aligned}
$$

Recalling that $\langle n_h \rangle_q = D_s$ and $\langle n_h \rangle_g = D_g$, we thus have

$$
r = \frac{r_+ D_s^+ + r_- D_s^-}{D_s^+ + D_s^-} = \frac{r_+ + r_- D_s^- / D_s^+}{1 + D_s^- / D_s^+}. \qquad (31)
$$

7. Analysis

We are now in a position to perform a global fit to the available experimental data of our formulas and to extract the nonperturbative constants $D_g(Q_0^2)$ and $D_s(Q_0^2)$.

We have to make a choice for the scale Q_0, which, in principle, is arbitrary. The perturbative series appears to be more rapidly converging at relatively large values of Q_0. Therefore, **we adopt $Q_0 = 50$ GeV in the following.**

We included the measurements of average gluon jet multiplicities and those of average quark jet multiplicities, which include 27 and 51 experimental data points, respectively. The errors correspond to 90% CL as explained above. All these fit results are in agreement with the experimental data.

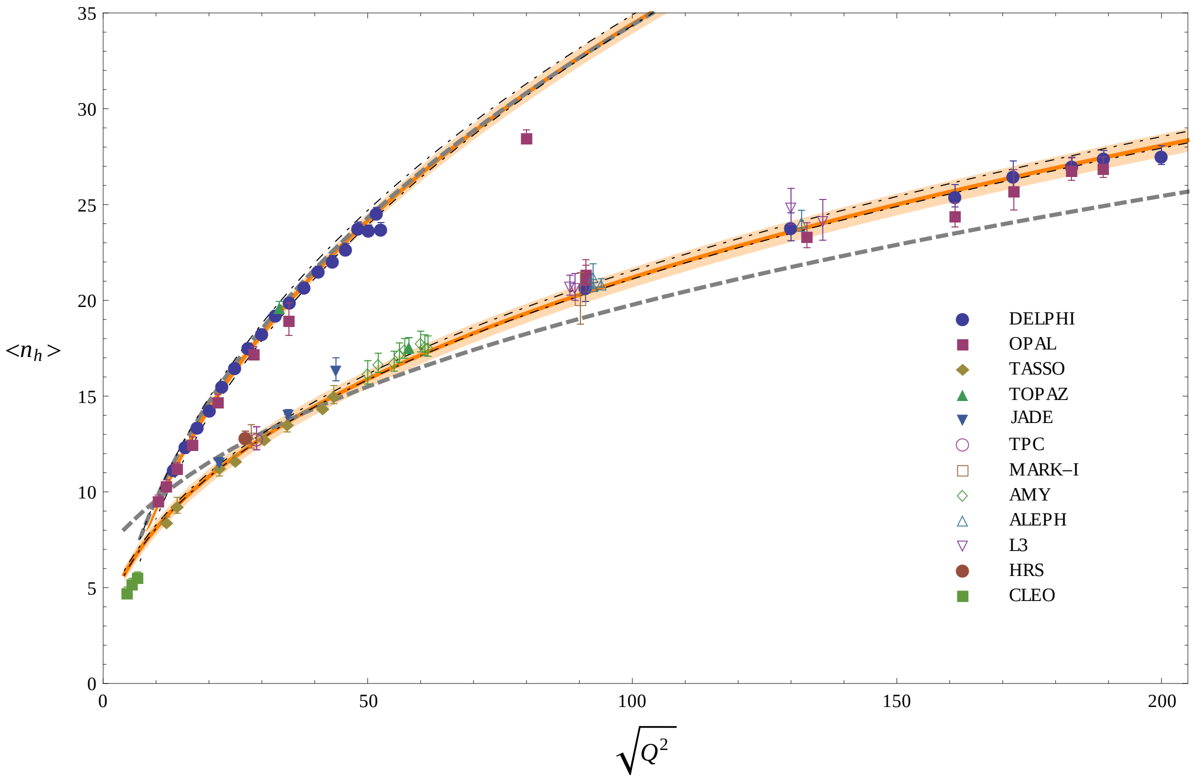

Figure 2: The average gluon (upper curves) and quark (lower curves) jet multiplicities evaluated, respectively, in the LO + NNLL (dashed/gray lines) and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ are compared with the experimental data included in the fits. The experimental and theoretical uncertainties in the $\mathrm{N^3LO_{approx+NLO+NNLL}}$ results are indicated by the shaded/orange bands and the bands enclosed between the dot-dashed curves, respectively.

In Fig. 2, we show as functions of Q the average gluon and quark jet multiplicities evaluated at $\mathrm{LO+NNLL}$ and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV.

- The fit of quark average multiplicity is good because minus component: there is the additional contribution with the additional free parameter $D_s(Q_0^2)$.
  The quark-singlet minus component comes with an arbitrary normalization and has a slow Q^2 dependence. Consequently, its numerical contribution may be approximately mimicked by a constant introduced to the average quark jet multiplicity as in (P.Abreu et al. [DELPHI Collab.], 1998)
- We can compare our results with the data for the ratio of the gluon and quark average multiplicities. It is not a fit because all our parameters have been already fixed.

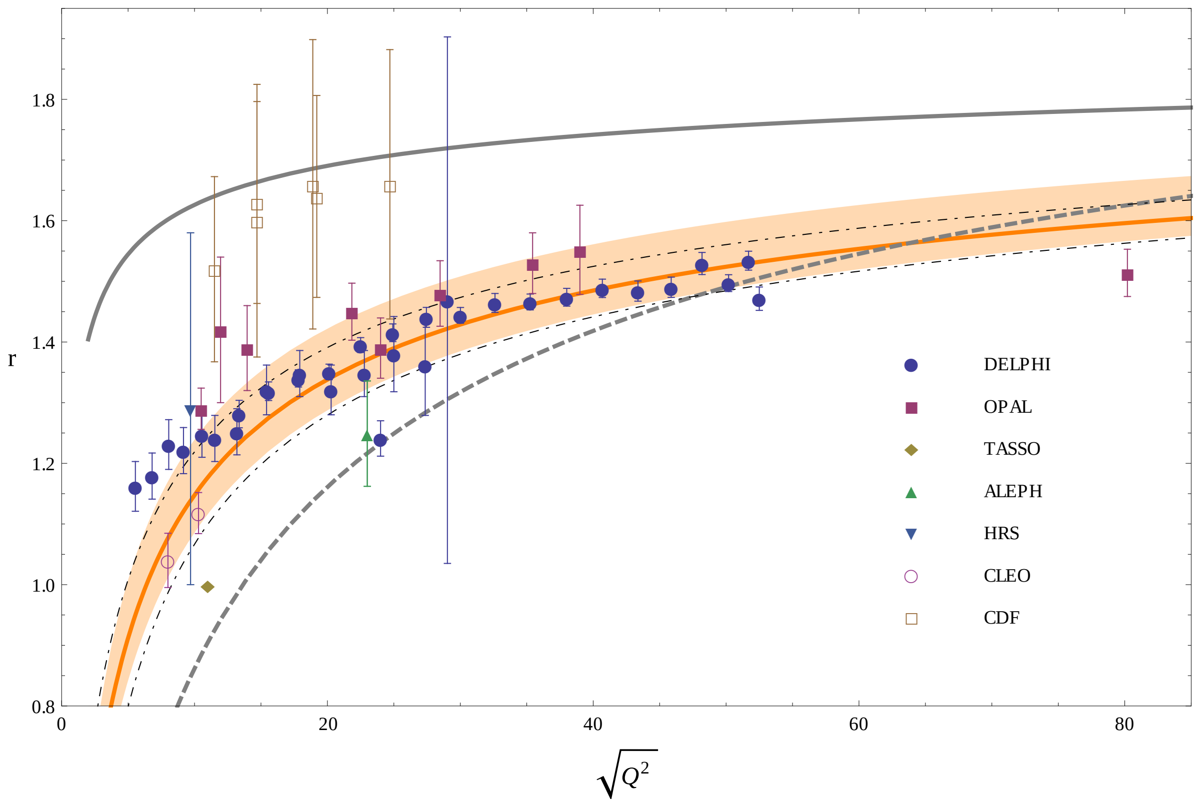

Figure 3: The average gluon-to-quark jet multiplicity ratio evaluated in the LO + NNLL (dashed/gray lines) and N^{3}I (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from Table **??** are com mental data. The experimental and theoretical uncertainties in the N^{3}LO$_{\text{approx+NLO+NNLL}}$ result are indicated by the shaded/or bands enclosed between the dot-dashed curves, respectively. The prediction given by analysis in (A.Capella, I.M.Dremin, J.W.G and J. Tran Thanh Van, 2000) is indicated by the continuous/gray line.

9. Conclusion

- **Prior to our 2013 analyses, experimental data on the average gluon and quark jet multiplicities could not be simultaneously described in a satisfactory way** mainly because the theoretical formalism failed to account for the difference in hadronic contents between gluon and quark jets, although the convergence of perturbation theory seemed to be well under control.

- **This problem was solved by including the minus components governed by $\hat{T}_{-}^{\mathrm{res}}(0, Q^2, Q_0^2)$.** The quark-singlet minus component comes with an arbitrary normalization and has a slow Q^2 dependence: **its numerical contribution may be approximately mimicked by a constant as it was in** (P.Abreu et al. [DELPHI Collab.], 1998).

- In our **2017** analysis, we have observed the new SUSY-like relation between the anomalous dimensions of the gluon and quark multiplicities.
- Motivated by the goodness of our fits with fixed value of $\alpha_s^{(5)}(m_Z^2)$ here, we then included $\alpha_s^{(5)}(m_Z^2)$ among the fit parameters, which yielded a further reduction of χ^2_{dof}. The obtained value $\alpha_s^{(5)}(m_Z^2) = 0.1205 \pm 0.0020$ is close to the world average one.

Next steps:

- To add $a_s^2 \sim \gamma_0^4$ correction to our analyses, which will be include N^{3}LL resummation, and to see the (type of the) possible violation the new SUSY-like relation between the anomalous dimensions.