The AVV triangle diagram in $SU(N_c)\ QCD$
and the generalized Crewther relation: scheme
(in)dependent results

Kataev A. L.
Molokoedov V. S.

INR RAS, Moscow and MIPT, Dolgoprudny

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Based on the results
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\[ G^{abc}_{\mu\nu\rho}(p, q) = i \int < 0 | T A^a_\mu(x) V^b_\nu(y) V^c_\rho(0) | 0 > e^{i(px+qy)} dx dy, \]

\[ V^a_\mu(x) = \bar{\psi}(x) \gamma_\mu t^a \psi(x) \] — vector non-singlet (NS) current,

\[ A^a_\mu(x) = \bar{\psi}(x) \gamma_\mu \gamma_5 t^a \psi(x) \] — axial NS current.

\[ \int T V^a_\mu(x) V^b_\nu(0) e^{iqx} dx \Big|_{q^2 \to \infty} \simeq d^{abc} \varepsilon_{\mu\nu\rho\lambda} \frac{q^\lambda}{Q^2} C_{Bjp} A^c_\rho(0) + \ldots , \]

\[ i \int T A^a_\mu(x) A^b_\nu(0) e^{iqx} dx = \delta^{ab} (g_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2) . \]
The Bjorken function $C_{Bjp}(a_s)$ is a characteristic of deep inelastic scattering of charged leptons on polarized nucleons and is related to the corresponding Bjorken polarized sum rule:

$$
\int_0^1 \left( g_1^{lp}(x, Q^2) - g_1^{ln}(x, Q^2) \right) dx = \frac{1}{6} \left| \frac{g_A}{g_V} \right| C_{Bjp}(a_s(Q^2)),
$$

where $g_1^{lp}$ and $g_1^{ln}$ are the structure functions of polarized lepton-proton and lepton-neutron deep inelastic scattering, characterizing the spin distribution of quarks inside nucleons. The ratio of axial and vector charge of the neutron $\beta$-decay is $g_A/g_V \approx -1.2723 \pm 0.0023$. 

Neglecting the mass dependence in the conditions of a large momentum transfer \( Q^2 \) in the \( \overline{\text{MS}} \) renormalization scheme in the \( \mathcal{O}(a_s^4) \) approximation of PT the Bjorken function can be represented in the form of two terms:

\[
C_{Bjp}(a_s) = C_{Bjp}^{NS}(a_s) + \sum_f Q_f C_{Bjp}^{SI}(a_s),
\]

where \( C_{Bjp}^{NS}(a_s) \) and \( C_{Bjp}^{SI}(a_s) \) — the flavor NS and SI contributions to the Bjorken function, \( a_s = \alpha_s/\pi \), \( Q_f \) is the electric charge of the \( f \)-th quark.

\[
C_{Bjp}^{NS}(a_s) = 1 + \sum_{k \geq 1} c_k a_s^k.
\]
Non-singlet contribution

The coefficients of this series are known up to the fourth order:

\[ c_1^{\overline{\text{MS}}} = -\frac{3}{4} C_F , \]

\[ c_2^{\overline{\text{MS}}} = \frac{21}{32} C_F^2 - \frac{23}{16} C_F C_A + \frac{1}{2} C_F T_F n_f , \quad (\text{Gorishny, Larin, 1986}) , \]

\[ c_3^{\overline{\text{MS}}} = -\frac{3}{128} C_F^3 + \left( \frac{1241}{576} - \frac{11}{12} \zeta_3 \right) C_F^2 C_A + \left( -\frac{5437}{864} + \frac{55}{24} \zeta_5 \right) C_F C_A^2 \]

\[ - \left( \frac{133}{576} + \frac{5}{12} \zeta_3 \right) C_F^2 T_F n_f + \left( \frac{3535}{864} + \frac{3}{4} \zeta_3 - \frac{5}{6} \zeta_5 \right) C_F C_A T_F n_f \]

\[ - \frac{115}{216} C_F T_F^2 n_f^2 , \quad (\text{Larin, Vermaseren, 1991}) , \]

where \( C_F \) and \( C_A \) are the Casimir operators, \( (T^a T^a)_{ij} = C_F \delta_{ij} \), \( f^{acd} f^{bcd} = C_A \delta^{ab} \), \( T_F = 1/2 \). For special case \( SU_c(3) \) QCD \( C_F = 4/3 \), \( C_A = 3 \). \( \zeta_n \) is the Riemann zeta-function. The coefficient \( c_4^{\overline{\text{MS}}} \) was calculated in the work (Baikov, Chetyrkin, Kühn, Phys. Rev. Lett. 104 (2010) 132004 ) and contains additional color configurations \( d_F^{abcd} d_A^{abcd} \) and \( d_F^{abcd} d_F^{abcd} \).
The Adler function

The Adler function is related to the experimentally measured characteristic of the electron-positron annihilation process into hadrons, called the $R$-relation

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \gamma \rightarrow \text{hadrons})}{\sigma_{\text{Born}}(e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-)},$$

by means of the dispersion relation:

$$D(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{(s + Q^2)^2} = -12\pi^2 Q^2 \frac{d}{dQ^2} \Pi(Q^2).$$

By analogy with the Bjorken function the following representation holds:

$$D(Q^2) = N_c \left( \sum_f Q_f^2 D^{NS}(a_s) + \left( \sum_f Q_f \right)^2 D^{SI}(a_s) \right).$$
NS contribution to the Adler function

\[ d_1^{\overline{\text{MS}}} = \frac{3}{4} C_F , \]

\[ d_2^{\overline{\text{MS}}} = -\frac{3}{32} C_F^2 + \left( \frac{123}{32} - \frac{11}{4} \zeta_3 \right) C_F C_A + \left( -\frac{11}{8} + \zeta_3 \right) C_F T_F n_f , \]

\[ (\text{Chetyrkin, Kataev, Tkachov, 1979}) \]

\[ d_3^{\overline{\text{MS}}} = -\frac{69}{128} C_F^3 + \left( -\frac{127}{64} - \frac{143}{16} \zeta_3 + \frac{55}{4} \zeta_5 \right) C_F^2 C_A + \]

\[ + \left( \frac{90445}{3456} - \frac{2737}{144} \zeta_3 - \frac{55}{24} \zeta_5 \right) C_F C_A^2 + \left( -\frac{29}{64} + \frac{19}{4} \zeta_3 - 5\zeta_5 \right) C_F^2 T_F n_f \]

\[ + \left( -\frac{485}{27} + \frac{112}{9} \zeta_3 + \frac{5}{6} \zeta_5 \right) C_F C_A T_F n_f + \left( \frac{151}{54} - \frac{19}{9} \zeta_3 \right) C_F T_F^2 n_f^2 \]

\[ (\text{Gorishnny, Kataev, Larin, 1991}) . \]

The coefficient \( d_4^{\overline{\text{MS}}} \) was computed in work of (Baikov, Chetyrkin, Kühn, Phys. Rev. Lett. 104 (2010) 132004 ) and confirmed later in work (Herzog, Ruijl, Ueda, Vermaseren, Vogt, JHEP 1708 (2017) 113).
On the other hand, in the conformally invariant limit this AVV three-point Green’s function is proportional to the triangular one-loop fermionic loop that determines the $\pi^0 \rightarrow \gamma\gamma$ decay:

$$G^{abc}_{\mu\nu\rho}(p, q)\bigg|_{\text{conf-inv}} = d^{abc}_{\mu\nu\rho} \Delta^{1-\text{loop}}_{\mu\nu\rho}(p, q).$$

In this limit in the massless QCD the Crewther relation is performed (Crewther, 1972):

$$D^{NS} C^{NS}_{Bjp} \bigg|_{c-i\ \text{limit}} = 1.$$
However, when the charge renormalization is taken into account, the conformal symmetry is violated. This circumstance leads to a modification of the Crewther relation:

$$D^{NS}(a_s)C_{B\beta\gamma}^{NS}(a_s) = 1 + \Delta_{csb}(a_s),$$

where term $\Delta_{csb}(a_s)$ can be presented in the gauge-invariant $\overline{\text{MS}}$-scheme in the following factorized form (Broadhurst, Kataev, Phys. Lett. B315 (1993)):

$$\Delta_{csb} = \left(\frac{\beta(a_s)}{a_s}\right) \sum_{i \geq 1} K_i a_s^i, \quad \beta(a_s) = \mu^2 \frac{d\alpha_s}{d\mu^2} = - \sum_{i \geq 0} \beta_i a_s^{i+2}.$$

This statement was confirmed explicitly at the $O(a_s^4)$ level by (Baikov, Chetyrkin, Kühn, Phys. Rev. Lett. 104 (2010) 132004). Theoretical indications on validity in all orders of PT were given in works of (Crewther (1997); Gabadadze, Kataev (1995)) and others.
Consideration in the mMOM-scheme

Is the $\beta(a_s)$-factorization possible in the \textbf{gauge-dependent renormalization schemes} such as the mMOM-scheme?

$$A_0^a, \mu = \sqrt{Z_A} A_\mu^a, \quad c_0^a = \sqrt{Z_c} c^a, \quad g_0 = \mu^\varepsilon Z_g g, \quad \xi_0 = Z_A Z_\xi^{-1} \xi,$$

where $A_\mu^a, c^a$ are fields of gluons and ghosts correspondingly, $\xi$ is the gauge parameter, included in the Lagrangian in the form $(\partial_\mu A_\mu^a)^2 / 2\xi$. The gauge-dependent mMOM-scheme is determined by the requirement of equality of the renormalization constant of the gluon-ghost-antighost vertex $Z_{cg} = Z_g Z_A^{1/2} Z_c$ to its analogue, defined in the $\overline{\text{MS}}$-scheme (Smekal, Maltman, Sternbeck, Phys. Lett. B681 (2009)):

$$Z_{cg}^{\text{mMOM}}(a_s^{\text{mMOM}}) = Z_{cg}^{\overline{\text{MS}}}(a_s^{\overline{\text{MS}}})$$

In this case the relation between coupling constants in these two schemes will look like as

$$a_s^{\text{mMOM}}(\mu^2) = \frac{Z_A^{\text{mMOM}}}{Z_A^{\overline{\text{MS}}} \left( \frac{Z_c^{\text{mMOM}}}{Z_c^{\overline{\text{MS}}}} \right)^2 a_s^{\overline{\text{MS}}}(\mu^2)}.$$
Taking into account the renormalization conditions on the polarization operators of gluons and ghosts, we obtain the following relations between the coupling constants and the gauge parameters (Gracey, J.Phys. A46 (2013) 225403), (Ruijl, Ueda, Vermaseren, Vogt, JHEP 1706 (2017) 040):

\[ a_{s\text{MOM}}(\mu^2) = \left(1 + \Pi_{\text{MS}}(\mu^2)\right)^{-1} \left(1 + \tilde{\Pi}_{\text{MS}}(\mu^2)\right)^{-2} a_{s\text{MS}}(\mu^2), \]

\[ \xi_{\text{MOM}}(\mu^2) = \left(1 + \Pi_{\text{MS}}(\mu^2)\right) \xi_{\text{MS}}, \]

where \( \Pi_{\text{MS}} \) and \( \tilde{\Pi}_{\text{MS}} \) are the self-energy operators of the gluons and ghosts correspondingly, which depend on \( \xi_{\text{MS}} \).
Further we find the following expansions of the quantities defined in the $\overline{\text{MS}}$-scheme in terms of the quantities computed in the mMOM-scheme:

$$
\xi^{\overline{\text{MS}}} = \xi \left( 1 + \left[ \left( \frac{97}{144} + \frac{1}{8} \xi + \frac{1}{16} \xi^2 \right) C_A - \frac{5}{9} T_{F n_f} \right] a_s + \\
+ \left[ \left( \frac{5591}{4608} - \frac{3}{16} \zeta_3 + \left( - \frac{121}{1536} + \frac{1}{8} \zeta_3 \right) \xi + \frac{7}{256} \xi^2 + \frac{7}{256} \xi^3 + \frac{1}{256} \xi^4 \right) C_A^2 + \\
+ \left( - \frac{371}{576} - \frac{1}{2} \zeta_3 \right) C_A T_{F n_f} + \left( - \frac{55}{48} + \zeta_3 \right) C_{F T F n_f} \right] a_s^2 + \ldots \right),
$$

$$
a_s^{\overline{\text{MS}}} = a_s + b_1 a_s^2 + b_2 a_s^3 + \ldots,
$$

$$
b_1 = \left( - \frac{169}{144} - \frac{1}{8} \xi - \frac{1}{16} \xi^2 \right) C_A + \frac{5}{9} T_{F n_f},
$$

$$
b_2 = \left( - \frac{18941}{20736} + \frac{39}{128} \zeta_3 + \left( \frac{889}{2304} - \frac{11}{64} \zeta_3 \right) \xi + \left( \frac{203}{2304} + \frac{3}{128} \zeta_3 \right) \xi^2 - \frac{3}{256} \xi^3 \right) C_A^2 \\
+ \left( - \frac{107}{648} + \frac{\zeta_3}{2} - \frac{5}{36} \xi - \frac{5}{72} \xi^2 \right) C_A T_{F n_f} + \left( \frac{55}{48} - \zeta_3 \right) C_{F T F n_f} + \frac{25}{81} T_{F n_f}^2.
$$

where $a_s^{\text{mMOM}} = a_s$ и $\xi^{\text{mMOM}} = \xi$. 
\[
\beta_{\text{mMOM}} = \beta_{\text{MS}} \frac{\partial a_s^{\text{mMOM}}}{a_s^{\text{MS}}} + \xi_{\text{MS}} \gamma_{\xi} \frac{\partial a_s^{\text{mMOM}}}{\partial \xi_{\text{MS}}},
\]

\[
\beta_{0}^{\text{mMOM}} = \frac{11}{12} C_A - \frac{1}{3} T_{F_f n_f},
\]

\[
\beta_{1}^{\text{mMOM}} = \left[ \frac{17}{24} - \frac{13}{192} \xi - \frac{5}{96} \xi^2 + \frac{1}{64} \xi^3 \right] C_A^2 + \left[ - \frac{5}{12} + \frac{1}{24} \xi + \frac{1}{24} \xi^2 \right] C_A T_{F_f n_f} - \frac{1}{4} C_F T_{F_f n_f},
\]

\[
\beta_{2}^{\text{mMOM}} = \left[ \frac{9655}{4608} - \frac{143}{512} \zeta_3 + \left( - \frac{1097}{6144} + \frac{33}{512} \zeta_3 \right) \xi + \left( - \frac{725}{6144} + \frac{13}{512} \zeta_3 \right) \xi^2 + \left( \frac{21}{2048} - \frac{3}{512} \right) \xi^3 + \frac{55}{6144} \xi^4 \right] C_A^3 + \left[ - \frac{2009}{1152} - \frac{137}{384} \zeta_3 + \frac{37}{384} \xi + \left( \frac{23}{256} - \frac{\zeta_3}{128} \right) \xi^2 + \left( \frac{1}{128} \right) \xi^3 \right] C_A C_F T_{F_f n_f} + \left[ - \frac{641}{576} + \frac{11}{12} \zeta_3 + \frac{1}{16} \xi + \frac{3}{64} \xi^2 \right] C_A C_F T_{F_f n_f} + \frac{1}{32} C_F^2 T_{F_f n_f} + \left[ \frac{23}{96} + \frac{\zeta_3}{6} \right] C_A T_{F_f n_f}^2 + \left[ \frac{23}{72} - \frac{\zeta_3}{3} \right] C_F T_{F_f n_f}^2.
\]
NS contribution to the Bjorken function in the mMOM-scheme

Using the renormalization invariance of the $C^\text{NS}_{Bjp}(a_s)$ function and the obtained relation $a_s^\text{MS} (\xi^\text{mMOM}, a_s^\text{mMOM})$ we find:

$$c_1^\text{mMOM} = -\frac{3}{4} C_F,$$

$$c_2^\text{mMOM} = \frac{21}{32} C_F^2 + \left( -\frac{107}{192} + \frac{3}{32} \xi + \frac{3}{64} \xi^2 \right) C_F C_A + \frac{1}{12} C_F T_F n_f,$$

$$c_3^\text{mMOM} = -\frac{3}{128} C_F^3 + \left[ \frac{13}{9} + \frac{3}{8} \zeta_3 - \frac{5}{6} \zeta_5 - \frac{1}{48} \xi - \frac{1}{96} \xi^2 \right] C_F C_A T_F n_f +$$

$$+ \left[ -\frac{13}{36} + \frac{\zeta_3}{3} \right] C_F^2 T_F n_f + \left[ \frac{1415}{2304} - \frac{11}{12} \zeta_3 - \frac{21}{128} \xi - \frac{21}{256} \xi^2 \right] C_F^2 C_A -$$

$$-\frac{5}{24} C_F T_F^2 n_f^2 + \left[ -\frac{20585}{9216} - \frac{117}{512} \zeta_3 + \frac{55}{24} \zeta_5 + \left( \frac{215}{3072} + \frac{33}{256} \zeta_3 \right) \xi +$$

$$\left( \frac{349}{3072} - \frac{9}{512} \zeta_3 \right) \xi^2 + \frac{9}{1024} \xi^3 \right] C_F C_A^2.$$
NS contribution to the Adler function in the mMOM-scheme

\[ d_{1\text{MOM}} = \frac{3}{4} C_F, \]

\[
d_{2\text{MOM}} = -\frac{3}{32} C_F^2 + \left[ \frac{569}{192} - \frac{11}{4} \zeta_3 - \frac{3}{32} \xi - \frac{3}{64} \xi^2 \right] C_F C_A + \left[ \zeta_3 - \frac{23}{24} \right] C_F T_F n_f, \]

\[
d_{3\text{MOM}} = -\frac{69}{128} C_F^3 + \left[ -\frac{1355}{768} - \frac{143}{16} \zeta_3 + \frac{55}{4} \zeta_5 + \frac{3}{128} \xi + \frac{3}{256} \xi^2 \right] C_F C_A +
\left[ -\frac{2033}{192} + \frac{89}{12} \zeta_3 + \frac{5}{6} \zeta_5 + \left( \frac{23}{96} - \frac{\zeta_3}{4} \right) \xi + \left( \frac{23}{192} - \frac{\zeta_3}{8} \right) \xi^2 \right] C_F C_A T_F n_f +
\left[ \frac{50575}{3072} - \frac{18929}{1536} \zeta_3 - \frac{55}{24} \zeta_5 + \left( -\frac{2063}{3072} + \frac{143}{256} \zeta_3 \right) \xi + \left( -\frac{1273}{3072} + \frac{185}{512} \zeta_3 \right) \xi^2 - \frac{9}{1024} \xi^3 \right] C_F C_A^2 +
\left[ \frac{29}{96} + 4 \zeta_3 - 5 \zeta_5 \right] C_F^2 T_F n_f + \left[ \frac{3}{2} - \zeta_3 \right] C_F T_F^2 n_f^2. \]
Is the factorization of RG $\beta$-function possible in the generalized Crewther relation in the gauge-dependent schemes?

Using the obtained expressions for the Bjorken and Adler functions, we find that in the $\mathcal{O}(a_s^2)$ approximation the factorization of the $\beta$-function is possible for any value of gauge parameter $\xi$:

$$K_1^{\text{mMOM}} = K_1^{\text{MS}} = \left(- \frac{21}{8} + 3\xi_3\right) C_F,$$

In the $\mathcal{O}(a_s^3)$ order of PT this property holds for three values of the gauge parameter only, namely

$$\xi = -3, \quad -1, \quad 0.$$
The $\beta$-factorization in the mMOM-scheme in the $O(a_s^3)$ approximation

Landau gauge $\xi = 0$:

$$K_2^{\text{mMOM}} = \left( \frac{397}{96} + \frac{17}{2} \zeta_3 - 15 \zeta_5 \right) C_F^2 + \left( - \frac{2591}{192} + \frac{91}{8} \zeta_3 \right) C_F C_A +$$

$$+ \left( \frac{31}{8} - 3 \zeta_3 \right) C_F T_F n_f ,$$

anti-Feynman gauge $\xi = -1$:

$$K_2^{\text{mMOM}} = \left( \frac{397}{96} + \frac{17}{2} \zeta_3 - 15 \zeta_5 \right) C_F^2 + \left( - \frac{1327}{96} + \frac{47}{4} \zeta_3 \right) C_F C_A +$$

$$+ \left( \frac{31}{8} - 3 \zeta_3 \right) C_F T_F n_f ,$$

Stefanis–Mikhailov gauge $\xi = -3$:

$$K_2^{\text{mMOM}} = \left( \frac{397}{96} + \frac{17}{2} \zeta_3 - 15 \zeta_5 \right) C_F^2 + \left( - \frac{695}{48} + \frac{25}{2} \zeta_3 \right) C_F C_A +$$

$$+ \left( \frac{31}{8} - 3 \zeta_3 \right) C_F T_F n_f .$$
The $\beta$-factorization in the mMOM-scheme in the $\mathcal{O}(\alpha_s^4)$ approximation

In the $\mathcal{O}(\alpha_s^4)$ order of PT the factorization property of the RG $\beta$-function in the generalized Crewther relation remains valid for the Landau gauge $\xi = 0$ only, namely

$$K_{3, \xi=0}^{\text{mMOM}} = \left( \frac{2471}{768} + \frac{61}{8} \zeta_3 - \frac{715}{8} \zeta_5 + \frac{315}{4} \zeta_7 \right) C_F^3 + \left( \frac{132421}{4608} + \frac{451}{8} \zeta_3 - \frac{3685}{48} \zeta_5 - \frac{105}{8} \zeta_7 \right) C_F^2 C_A +$$

$$+ \left( - \frac{1840145}{18432} + \frac{152329}{3072} \zeta_3 + \frac{2975}{48} \zeta_5 - \frac{2113}{128} \zeta_3^2 \right) C_F C_A^2 +$$

$$+ \left( - \frac{1273}{144} - \frac{599}{24} \zeta_3 + \frac{75}{2} \zeta_5 \right) C_F^2 T_F n_f + \left( - \frac{49}{6} + \frac{7}{2} \zeta_3 + 5 \zeta_5 \right) C_F T_F^2 n_f^2 +$$

$$\left( \frac{71251}{1152} - \frac{539}{24} \zeta_3 - \frac{125}{3} \zeta_5 + \frac{5}{2} \zeta_3^2 \right) C_F C_A T_F n_f.$$
Consideration in the MOMggg-scheme

It is interesting to find out whether there are other MOM-schemes in QCD, which respect the property of the β-function factorization in the GCR for concrete choice of the gauge parameter. We consider the MOMggg-scheme, determined by renormalization of the quartic gluon vertex through subtractions of UV divergences in the symmetric point (Gracey, Phys. Rev. D 90 (2014) 025011). For Landau gauge in QCD with SU(3) color group we find:

$$\beta_1^{\text{MOMggg}} \bigg|_{\xi=0}^{N_c=3} = \frac{51}{8} - \frac{19}{24} n_f,$$

$$K_2^{\text{MOMggg}} \bigg|_{\xi=0}^{N_c=3} = -\frac{280073}{8640} + \frac{3017}{100} \log \left( \frac{4}{3} \right) - \frac{595}{256} \Phi_1 - \frac{50533}{51200} \Phi_2$$

$$+ \zeta_3 \left( \frac{15973}{360} - \frac{862}{25} \log \left( \frac{4}{3} \right) + \frac{85}{32} \Phi_1 + \frac{7219}{6400} \Phi_2 \right) - \frac{80}{3} \zeta_5 +$$

$$\left[ \frac{65}{36} - \frac{49}{24} \log \left( \frac{4}{3} \right) - \frac{49}{96} \Phi_1 + \frac{7}{96} \Phi_2 + \zeta_3 \left( -\frac{10}{9} + \frac{7}{3} \log \left( \frac{4}{3} \right) + \frac{7}{12} \Phi_1 + \frac{1}{12} \Phi_2 \right) \right] n_f.$$
The special functions $\Phi_1$ and $\Phi_2$ are expressed through the Clausen function $\text{Cl}_2(\theta)$ and have the following form

$$\Phi_1 = \sqrt{2} \left[ 2 \text{Cl}_2 \left( 2 \arccos \left( \frac{1}{\sqrt{3}} \right) \right) + \text{Cl}_2 \left( 2 \arccos \left( \frac{1}{3} \right) \right) \right],$$

$$\Phi_2 = \frac{4}{\sqrt{5}} \left[ 2 \text{Cl}_2 \left( 2 \arccos \left( \frac{2}{3} \right) \right) + \text{Cl}_2 \left( 2 \arccos \left( \frac{1}{9} \right) \right) \right],$$

$$\text{Cl}_2(\theta) = - \int_{0}^{\theta} dx \log \left| 2 \sin \frac{x}{2} \right|,$$

and numerically $\Phi_1 \approx 2.832045$ and $\Phi_2 \approx 3.403614$ correspondingly.
Consideration in the $\text{MOM}gggg$-scheme

At $\xi = -3$ we have:

$$\frac{\beta_{1, \xi=-3}^{\overline{\text{MS}}}-\beta_{1, \xi=-3}^{\text{MOM}gggg}}{\beta_0} = -\frac{333}{20} - \frac{3537}{200} \log \left( \frac{4}{3} \right) + \frac{9}{4} \Phi_1 + \frac{33993}{6400} \Phi_2,$$

$$K_2^{\text{MOM}gggg}\Bigg|_{\xi=-3}^{N_c=3} = -\frac{9337}{270} + \frac{13769}{400} \log \left( \frac{4}{3} \right) - \frac{35}{32} \Phi_1 - \frac{2191}{12800} \Phi_2$$

$$+ \zeta_3 \left( \frac{2108}{45} - \frac{1967}{50} \log \left( \frac{4}{3} \right) + \frac{5}{4} \Phi_1 + \frac{313}{1600} \Phi_2 \right) - \frac{80}{3} \zeta_5$$

$$+ \left[ \frac{65}{36} - \frac{49}{24} \log \left( \frac{4}{3} \right) - \frac{49}{96} \Phi_1 + \frac{7}{96} \Phi_2 \right] n_f.$$

Thus, we come to conclusion that in the $\mathcal{O}(a_s^3)$ order of PT in the $\text{MOM}gggg$-scheme the factorization property holds in Landau and Stefanis–Mikhailov gauges and is not satisfied in anti-Feynman gauge (the gauge $\xi = -1$ is the feature of the mMOM-scheme since $\beta_1^{\text{mMOM}} = \beta_1^{\overline{\text{MS}}}$ in this gauge).
Are the values of $\xi = -3$ and $\xi = 0$ distinguished in all gauge-dependent schemes?

Using relation $a_{s}^{\overline{\text{MS}}} = a_{s}^{\text{AS}} + \sum_{k=1} b_{k}^{\text{AS}} (a_{s}^{\text{AS}})^{k+1}$ and explicit form of the term which breaks the conformal symmetry, we arrive at equation ($\text{AS}$ denotes any scheme with linear covariant gauge):

$$
\frac{\beta^{\overline{\text{MS}}}(a_{s}^{\overline{\text{MS}}}(a_{s}^{\text{AS}}))}{a_{s}^{\overline{\text{MS}}}(a_{s}^{\text{AS}})} K^{\overline{\text{MS}}}(a_{s}^{\overline{\text{MS}}}(a_{s}^{\text{AS}})) = \frac{\beta^{\text{AS}}(a_{s}^{\text{AS}})}{a_{s}^{\text{AS}}} K^{\text{AS}}(a_{s}^{\text{AS}}),
$$

which allows us to obtain the following relations:

$$
K_{1}^{\text{AS}} = K_{1}^{\overline{\text{MS}}},
$$

$$
K_{2}^{\text{AS}} = K_{2}^{\overline{\text{MS}}} + \left(\frac{\beta_{1}^{\overline{\text{MS}}} - \beta_{1}^{\text{AS}}}{\beta_{0}} + 2b_{1}^{\text{AS}}\right) K_{1}^{\overline{\text{MS}}},
$$

$$
K_{3}^{\text{AS}} = K_{3}^{\overline{\text{MS}}} + \left(\frac{\beta_{1}^{\overline{\text{MS}}} - \beta_{1}^{\text{AS}}}{\beta_{0}} + 3b_{1}^{\text{AS}}\right) K_{2}^{\overline{\text{MS}}} + \left(2b_{2}^{\text{AS}} + (b_{1}^{\text{AS}})^{2}\right.
$$

$$
\left. + \frac{\beta_{2}^{\overline{\text{MS}}} - \beta_{2}^{\text{AS}}}{\beta_{0}} + \frac{3\beta_{1}^{\overline{\text{MS}}} - 2\beta_{1}^{\text{AS}}}{\beta_{0}} b_{1}^{\text{AS}} + \frac{\beta_{1}^{\text{AS}}(\beta_{1}^{\text{AS}} - \beta_{0})}{\beta_{0}}\right) K_{1}^{\overline{\text{MS}}}. 
$$
Thus, we come to the conclusion that the question of the factorization of the $\beta$-function reduces to the conditions of division without remainder of terms of type $\left( \beta_{1}^{\overline{\text{MS}}} - \beta_{1}^{\text{AS}} \right) / \beta_{0}$, $\left( \beta_{2}^{\overline{\text{MS}}} - \beta_{2}^{\text{AS}} \right) / \beta_{0}$, $\beta_{1}^{\text{AS}} \left( \beta_{1}^{\text{AS}} - \beta_{1}^{\overline{\text{MS}}} \right) / \beta_{0}^{2}$ etc. (hence we conclude that there is the factorization in QED in all popular schemes, such as MS–like, MOM and OS-schemes).

Further we find the relation of the $\mathcal{O}(a_{s}^{2})$ coefficients of the $\beta$-functions:

$$\beta_{1}^{\overline{\text{MS}}} - \beta_{1}^{\text{AS}} = \xi \gamma_{0}^{\overline{\text{MS}}} (\xi) \frac{\partial b_{1}(\xi)}{\partial \xi},$$

where $\gamma_{0}^{\overline{\text{MS}}} = (-13/24 + \xi^{\overline{\text{MS}}}/8)C_{A} + T_{F}n_{f}/3$. From this relation we obtain, that at $\xi = 0$ $\beta_{1}^{\text{AS}} = \beta_{1}^{\overline{\text{MS}}}$, and division by $\beta_{0}$ is performed in the $\mathcal{O}(a_{s}^{3})$ approximation.

At $\xi = -3$: $\gamma_{0}^{\overline{\text{MS}}} = -\beta_{0}$ and division without remainder also carried out.
Similarly, we obtain the following formulas at $\xi = 0$:

\[
K_{2}^{AS} = K_{2}^{\overline{MS}} + 2b_{1}^{AS}(0)K_{1}^{\overline{MS}},
\]

\[
K_{3}^{AS} = K_{3}^{\overline{MS}} + 3b_{1}(0)K_{2}^{\overline{MS}} + 3b_{2}(0)K_{1}^{\overline{MS}},
\]

\[
K_{4}^{AS} = K_{4}^{\overline{MS}} + 4b_{1}(0)K_{3}^{\overline{MS}} + \left(4b_{2}(0) + 2b_{1}^{2}(0)\right)K_{2}^{\overline{MS}} + 4b_{3}(0)K_{1}^{\overline{MS}},
\]

\[
K_{5}^{AS} = K_{5}^{\overline{MS}} + 5b_{1}(0)K_{4}^{\overline{MS}} + 5\left(b_{2}(0) + b_{1}^{2}(0)\right)K_{3}^{\overline{MS}}
\]

\[
+ 5\left(b_{3}(0) + b_{1}(0)b_{2}(0)\right)K_{2}^{\overline{MS}} + 5b_{4}(0)K_{1}^{\overline{MS}} \ldots
\]

Thus, we come to the conclusion that the gauge invariance of the renormalization schemes is a sufficient condition for the factorization of the RG $\beta$-function in the GCR, but is not a necessary condition.
Conclusion

- Initially, it was assumed that the true cause of the $\beta$-factorization lies in the gauge invariance of the renormalization schemes. In a more detailed analysis, it was unexpectedly found that a similar property is possible in gauge-dependent schemes.
- On the example of the mMOM-scheme we explained that gauges $\xi = 0, -3$ are highlighted among the remaining values of $\xi$ (the gauge $\xi = -1$ is a specific of the mMOM-scheme).
- It is shown that for $\xi = -3$ the factorization of the $\beta$-function takes place in all schemes with linear covariant gauge in the $\mathcal{O}(a_s^3)$ approximation.
- The factorization in the Landau gauge occurs in all orders of PT (if such is observed in the $\overline{\text{MS}}$-scheme; there are theoretical grounds for believing this).
- The gauge invariance of renormalization schemes is a sufficient but not necessary condition for factorization.
- The question of the theoretical reason for the factorization of the $\beta$-function still remains open.
Thank you for your attention!