Special geometry on Calabi-Yau moduli spaces and Q-invariant Frobenius rings.

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Introduction.

When compactifying the Superstring theory on a Calabi–Yau (CY) threefold X, the low-energy effective theory is defined in terms of the Special Kähler geometry of the CY moduli space.

It is known that the Kähler potential given by the logarithm the holomorphic volume of Calabi-Yau manifold X_ϕ:

$$G(\phi)_{a\bar{b}} = \partial_a \overline{\partial_b} K(\phi, \bar{\phi}),$$
$$e^{-K(\phi)} = \int_{X_\phi} \Omega \wedge \overline{\Omega},$$

This can be rewritten in terms of periods of Ω as:

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega, \quad q_\mu \in H_3(X, \mathbb{R}).$$
$$e^{-K} = \omega_\mu(\phi) C_{\mu\nu} \overline{\omega_\nu(\phi)},$$

where $C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is an intersection matrix of 3-cycles.

New approach

In practice, computation of periods in the symplectic basis is a very complicated problem and was done explicitly only in few examples.

I'l present a new method to easily compute the Kähler metric for the large class of CY defined as hypersurfaces in weighted projective spaces.

The method uses the Correspondence between the Hodge structure on the middle cohomology of CY manifolds and the structure of the Invariant Frobenius Ring associated with CY manifolds.

This correspondence is realized by Oscillatory integral presentation for the periods of the holomorphic Calabi-Yau 3-form.

Trying to clarify this correspondence we obtain the very efficient method for computing Special geometry on the Moduli space.

Correspondence of the Hodge structure of $H^3(X)$ and R^Q.

Let X CY manifold realized as the zero-set of a quasi-homogeneous polynomial $W(x)$ in weighted P^4. Cohomology $H^3(X)$ with Hodge decomposition $H^3(X) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$, the complex conjugation and Poincare pairing is isomorphic to the invariant Milnor ring R^Q defined by $W(x)$ with its Hodge decomposition given by the degree grading, antiholomorphic involution M and the residue pairing $\eta_{\mu\lambda}$.

From this fact we get the formula for Kähler potential $K(\phi)$

$$e^{-K(\phi)} = \sigma_\mu(\phi)\,\eta_{\mu\lambda}\,M_{\lambda\nu}\,\overline{\sigma_\nu(\phi)}.$$

$\sigma_\mu(\phi)$ are periods computed as oscillatory integrals,

$\eta_{\mu\nu}$ is a residue pairing in the Milnor ring,

$M_{\mu\nu}$ is the antiholomorphic involution of the ring R^Q.

All the three $\sigma_\mu(\phi)$, $\eta_{\mu\nu}$, $M_{\mu\nu}$ can be efficiently computed.

Example. 101-d moduli space of Quintic threefold

Quintic CY manifold X be given as a solution of the equation

$$W(x,\phi)=\sum_{i=1}^{5}x_i^5+\sum_{\mathbf{s}=\mathbf{1}}^{101}\phi_{\mathbf{s}}\prod_i x_i^{s_i}=0$$

$\mathbf{s}$=(s_1,s_2,s_3,s_4,s_5), $0\le s_i\le 3$, $\deg(\mathbf{s}):=\sum_{i=1}^{5}s_i=5$.
The complex structures Kähler potential in this case is

$$e^{-K(\phi)}=\sum_{\mu=0}^{203}(-1)^{\deg(\mu)/5}\prod\gamma\left(\frac{\mu_i+1}{5}\right)|\sigma_\mu(\phi)|^2,$$

$$\sigma_\mu(\phi)=\sum_{n_1,\dots,n_5\ge 0}\prod_{i=1}^{5}\frac{\Gamma(\frac{\mu_i+1}{5}+n_i)}{\Gamma(\frac{\mu_i+1}{5})}\sum_{m\in\Sigma_n}\prod_s\frac{\phi_s^{m_s}}{m_s!},$$

μ=$(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$, $0\le\mu_i\le 3$, $\quad\sum_{i=1}^{5}\mu_i=0,5,10,15.$

$$\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)},\qquad \Sigma_n=\{m_{\mathbf{s}}\mid\sum_s m_{\mathbf{s}}s_i=5n_i+\mu_i\}$$

CY as the hypersurface in the weighted projective space

Let $x_1, \ldots, x_5$ be homogeneous coordinates in the weighted projective space $\mathbb{P}^4_{(k_1,\ldots,k_5)}$ and Calabi-Yau X defined as

$$X = \{x_1, \ldots, x_5 \ \in \mathbb{P}^4_{(k_1,\ldots,k_5)} |\ W_0(x) = 0\}.$$

For some quasi-homogeneous polynomial $W_0(x)$,

$$W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x_i)$$

and

$$\deg W_0(x) = d = \sum_{i=1}^{5} k_i.$$

The last relation ensures that X is a CY manifold.
The moduli space of complex structures is then given by homogeneous polynomial deformations of this singularity:

$$W(x, \phi) = W_0(x) + \sum_{s=0}^{h_{21}-1} \phi_{\mathbf{s}} e_s(x),$$

where $e_s(x)$ are monomials of x which have the same degree d.

Hodge structure on middle cohomology

The holomorphic everywhere non-vanishing 3-form Ω is defined as

$$\Omega = \frac{x_5 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial W(x)/\partial x_4}$$

Periods of Ω, needed for our goal are integrals over cycles of $H_3(X, \mathbb{R})$

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega, \quad q_\mu \in H_3(X, \mathbb{R}).$$

$H^3(X)$ possesses Hodge structure $H^3(X) = \oplus_{k=0}^{3} H^{3-k,k}(X)$,

$\dim H^{3,0}(X) = \dim H^{0,3}(X) = 1, \ \dim H^{2,1}(X) = \dim H^{1,2}(X) = h^{2,1}$.

Poincaré pairing can be written through integrals over cycles q_μ as

$$\eta(\chi_a, \chi_b) = \int_X \chi_a \wedge \chi_b = \int_{q_\mu} \chi_a \; C_{\mu\nu} \int_{q_\nu} \chi_b,$$

is invariant with respect to complex conjugation (p, q)-forms.
$C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is the intersection matrix of 3-cycles.

On the other hand the polynomial $W_0(x)$ defines a Milnor ring R_0. We consider its subring R^Q invariant in respect to the symmetry group Q, that acts on $\mathbb{C}^5$ diagonally and preserves $W(x,\phi)$

$$R^Q = \left(\frac{\mathbb{C}[x_1, \cdots, x_5]}{\mathrm{Jac}(W_0)} \right)^Q, \quad \mathrm{Jac}(W_0) = \langle \partial_i W_0 \rangle_{i=1}^5 .$$

R^Q becomes Frobenius ring if it endowed with pairing

$$\eta(e_\alpha, e_\beta) = \mathrm{Res} \, \frac{e_\alpha(x) e_\beta(x) \, \mathrm{d}^5 x}{\prod_{i=1}^N \partial_i W_0(x)} .$$

$\dim R^Q = \dim H^3(X)$ and R^Q has the Hodge structure in correspondence with degrees $0, d, 2d, 3d$ of its elements

$$R^Q = (R^Q)^0 \oplus (R^Q)^1 \oplus (R^Q)^2 \oplus (R^Q)^3$$

. $\dim(R^Q)^0 = \dim(R^Q)^3 = 1, \;\; \dim(R^Q)^1 = \dim(R^Q)^2 = h^{2,1}$

$Q-$invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$

By the next step we define two differentials $D_\pm$

$$D_\pm = \mathrm{d} \pm \mathrm{d}W_0\wedge, \qquad (D_\pm)^2 = 0$$

and two groups of $Q-$invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_Q$.
These groups inherit the grading degree structure from R^Q.
Choosing in the ring R^Q some basis $\{e_\mu(x)\}$ we take $\{e_\mu(x)\,\mathrm{d}^5x\}$ as a basis of $H^5_{D_\pm}(\mathbb{C}^5)_Q$.

These cohomology groups are in one-to-one correspondence with the middle cohomology group $\in H^3(X)$(Candelas 1988).

This isomorphism, defined below, maps the components $H^{3-q,q}(X)$ to the Hodge decomposition components of $H^5_\pm(\mathbb{C}^5)_Q$ spanned by $e_\mu(x)\,\mathrm{d}^5x$ with $e_\mu(x) \in (R^Q)^q$.

It also maps the Poincare pairing of differential forms to X to the pairing $\eta(e_\alpha, e_\beta)$ on the invariant ring R^Q.

Q-invariant relative homology and oscilatory integrals

Having $H^5_{D_\pm}(\mathbb{C}^5)_Q$ we define two Q-invariant the relative homology groups $\mathcal{H}_5^{\pm,Q} := H_5(\mathbb{C}^5,\ \mathrm{Re}\,W_0 = L \to \pm\infty)_Q$ as a quotient of the relative homology group $H_5(\mathbb{C}^5,\ \mathrm{Re}\,W_0 = L \to \pm\infty)$.

For this purpose we define the pairing via oscillatory integrals

$$\langle Q^\pm_\mu, e_\nu(x)\,\mathrm{d}^5x\rangle := \int_{Q^\pm_\mu} e_\nu(x)\, e^{\mp W(x)}\mathrm{d}^5x.$$

Using this pairing we define the relative invariant homology groups $\mathcal{H}_5^{\pm,Q}$ to be the quotient of $H_5(\mathbb{C}^5, W_0 = L,\ \mathrm{Re}L \to \pm\infty)$
by its subspace whose elements are ortogonal to
all elements of $H^5_{D_\pm}(\mathbb{C}^5)_Q$.

$H_3(X)$ versus R^Q correspondence

The crucial fact for further is that R^Q and $H^3(X)$ and all their additional structures on these rings are isomorhic to each other.

First of all there exists an isomorphism S of cycles for each ϕ this gives

$$S(Q_\mu^+) = q_\mu, \quad Q_\mu^+ \in \mathcal{H}_5^{\pm,Q}, \;\; q_\mu \in H_3(X,\mathbb{Z}).$$

The isomorphism is defined by the oscilatory integrals as follows. Let $\{q_\mu\}$ is a basis of $H_3(X,\mathbb{Z})$, then the basis $Q_\mu^\pm$ of $\mathcal{H}_5^{\pm,Q}$ can be choosen in such a way that the integrals over the corresponding cycles of these bases are equal

$$\int_{q_\mu} \Omega_\phi = \int_{Q_\mu^\pm} e^{\mp W(x,\phi)}\, \mathrm{d}^5 x.$$

$H^3(X)$ versus $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$ correspondence

Having isomorphism between $H_3(X)$ and $\mathcal{H}_5^{\pm,Q}$ we define the isomorphism between the two cohomology groups $H^3(X)$ and $H^5_{D_\pm}(\mathbb{C}^5)_Q$ also with help of oscilatory integrals.

Namely, take the basis of cycles $q_\mu \in H_3(X)$ and the corresponding to them basis of cycles $Q^\pm_\mu \in \mathcal{H}_5^{\pm,Q}$ at $\phi = 0$, then the form $\chi_\alpha \in H^3(X)$ corresponds to the form $e_\alpha(x)\,\mathrm{d}^5x \in H^5_{D_\pm}(\mathbb{C}^5)_Q$ if

$$\int_{q_\mu} \chi_\alpha = \int_{Q^\pm_\mu} e_\alpha(x)\, e^{\mp W(x,\phi)}\,\mathrm{d}^5x$$

for all pairs $\{q_\mu, Q_\mu\}$.
So these two forms are isomorphic if they have the equal coordinates (that is, periods) in some isomorphic bases.

This isomorphism preserves Hodge decomposition (Candelas).
The pairing of the form $\in H^3(X)$ and the pairing of the elements $\in R^Q$ coincide.

Coincidence of two pairings

To show this, we rewrite the Poincaré pairing of χ_a, χ_b in $H^3(X)$,

$$< \chi_a, \chi_b >:= \int_X \chi_a \wedge \chi_b$$

as as the bilinear expression of periods

$$< \chi_a, \chi_b >= \int_{q_\mu} \chi_a \; C_{\mu\nu} \int_{q_\nu} \chi_b,$$

where $C_{\mu\nu} = q_\mu \cap q_\nu$ is the intersection matrix of the cycles.
On the other hand the residue pairing $\eta(e_a, e_b)$ in the ring R^Q can be written through the periods (Chiodo et al) as:

$$\eta(e_a, e_b) = \int_{Q^+_\mu} e_a e^{-W(x,\phi)} \mathrm{d}^5 x \; C_{\mu\nu} \int_{Q^-_\mu} e_b e^{W(x,\phi)} \mathrm{d}^5 x,$$

where $C_{\mu\nu} = L^+_\mu \cap L^-_\nu$ is the same as above.
Comparing two expressions from the eguality for periods we obtain

$$< \chi_a, \chi_b >= \eta(e_a, e_b).$$

We will use the relation between $\eta(e_a, e_b)$ and $C_{\mu\nu}$ below for explicitly finding the intersection matrix in terms of R^Q pairing.

Anti-Involution $*$ on R^Q

The same isomorphism allows to define a complex conjugation $*$ on the Q-invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$.

Let the form $\phi_\mu \in H^3(X)$ corresponds to $\{e_\mu(x)\} \in R^Q$ and let

$$*\phi_\mu = M_{\nu\mu}\phi_\nu$$

then R^Q inherits this involution.

For the basis $\{e_\mu(x)\}$ the antiholomorphic operation $*$ reads as

$$* \, e_\mu(x) = M_{\nu\mu} e_\nu(x).$$

From this definition and since $(*)^2 = I$, it follows that $\bar{M}M = I$.

It is convenient to introduce the special basis $\Gamma^\pm_\mu$ dual to the basis $\{e_\mu(x)\}$ such that:

$$\langle \Gamma^\pm_\mu, e_\nu(x) \, \mathrm{d}^5 x \rangle = \int_{\Gamma^\pm_\mu} e_\nu(x) \, e^{\mp W_0(x)} \mathrm{d}^5 x = \delta_{\mu\nu}.$$

This definition induces the antiholomorphic operation $*$ on $\Gamma^\pm_\mu$

$$*\Gamma^\pm_\mu = \bar{M}_{\mu\nu}\Gamma^\pm_\nu$$

The cycles $\Gamma^{\pm}_{\mu}$ belongs to homology group $\mathcal{H}^{\pm,Q}_{5}$ with complex coefficients.
If we define T as a transition matrix from cycles $\Gamma^{\pm}_{\mu}$ to an arbitrary real basis of cycles, say, Lefschetz thimbles $L^{\pm}_{\mu} = *L^{\pm}_{\mu}$

$$L^{\pm}_{\mu} = T_{\mu\nu}\Gamma^{\pm}_{\nu}.$$

Then we have

$$L^{\pm}_{\mu} = \overline{T}_{\mu\nu}\,(*\Gamma^{\pm}_{\nu}).$$

Comparing this relation with

$$*\Gamma^{\pm}_{\mu} = \bar{M}_{\mu\nu}\,\Gamma^{\pm}_{\nu},$$

we obtain for M the useful expression in terms T

$$M = T^{-1}\bar{T}.$$

Obviously M does not depend on the choice of real cycles.
Using the definition $\langle \Gamma^{\pm}_{\mu}, e_{\nu}(x)\,\mathrm{d}^5x\rangle = \delta_{\mu\nu}$ we obtain the useful (as we will see) for computing $T_{\mu\nu}$ (and $M_{\mu\nu}$) relation

$$T_{\mu\nu} = \int_{L^{\pm}_{\mu}} e_{\nu}(x)\, e^{\mp W_0(x)}\mathrm{d}^5x.$$

Deriving the main formula for Kähler potential

Now use the CY/R^Q correspondence to transform the expression

$$e^{-K} = \omega_b^+(\phi)\ C_{ab}\ \overline{\omega_b^-(\phi)}$$

where periods given by oscilatory integrals over cycles $L_a^\pm$

$$\omega_a^\pm(\phi) = \int_{L_a^\pm} e^{\mp W(x,\phi)} \mathrm{d}^5 x = T_{a\mu}\sigma_\mu^\pm(\phi),$$

and periods $\sigma_\mu^\pm(\phi)$ are integrals over cycles $\Gamma_\mu^\pm$

$$\sigma_\mu^\pm(\phi) = \int_{\Gamma_\mu^\pm} e^{\mp W(x,\phi)} \mathrm{d}^5 x.$$

We have also the expression for pairing on R^Q through the same periods $\omega_a^\pm(\phi)$ in $\phi = 0$

$$\eta(e_\mu, e_\nu) = \int_{L_a^+} e_\mu e^{-W_0(x)} \mathrm{d}^5 x\ C_{ab} \int_{L_b^-} e_\nu e^{W_0(x)} \mathrm{d}^5 x = T_{a\mu} C_{ab} T_{b\nu}$$

Excluding from these relations the matrix C_{ab} we obtain finally

$$e^{-K(\phi)} = \sum_{\mu,\nu,\lambda} \sigma_\mu^+(\phi)\ \eta_{\mu\lambda}\ M_{\lambda\nu}\ \overline{\sigma_\nu^-(\phi)}.$$

Example. Quintic threefold

In this case CY manifold X is given by the equation

$$X = \{(x_1 : \cdots : x_5) \in \mathbb{P}^4 \mid W(x,\phi) = 0\},$$

$$W(x,\phi) = W_0(x) + \sum_{t=0}^{100} \phi_t e_t(x), \; W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$$

and $e_t(x)$ are the degree 5 monomials such that each variable has the power that is a non-negative integer less then four.
Here monomials $e_t(x) = x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5}$, ($t := (t_1, \cdots, t_5)$).

There are precisely 101 of such monomials, which can be divided into 5 sets in respect to the permutation group S_5: $(1,1,1,1,1)$, $(2,1,1,1,0)$, $(2,2,1,0,0)$, $(3,1,1,0,0)$, $(3,2,0,0,0)$.
In these groups there are 1, 20, 30, 30, 20 different monomials.

We denote $e_0(x) := e_{(1,1,1,1,1)}(x) = x_1x_2x_3x_4x_5$ to be the so-called fundamental monomial, which will be somewhat distinguished in our picture. For Quintic $\dim H_3(X) = 204$.

Q-nvariant Ring

We can consider $W_0(x)$ as an isolated singularity in $\mathbb{C}^5$.
Then we have an associated Milnor ring

$$R_0 = \frac{\mathbb{C}[x_1, \cdots, x_5]}{\langle \partial_i W \rangle}.$$

For Quintic threefold X its Milnor ring R_0 is generated as a vector space by monomials where each variable has degree less than four, and its dim $R_0 = 1024$.

Polynomial $W_0(x)$ is homogeneous and, in particular,
$W_0(\alpha x_1, \ldots, \alpha x_5) = W_0(x_1, \ldots, x_5)$ for $\alpha^5 = 1$.
This action preserves $W(x)$ and is trivial in the corresponding projective space and on X.

Such a group with this action is called the quantum symmetry Q.
In this case $Q \simeq \mathbb{Z}_5$.

Milnor Ring

Quantum group Q obviously acts on the Milnor ring R_0.
We define a subring R^Q in the Milnor ring R_0,

$$R^Q := \{e_\mu(x) \in R_0 \mid e_\mu(\alpha x) = e_\mu(x)\},\ \alpha^5 = 1,$$

to be a Q-invariant part of the Milnor ring. $\dim R^Q = 204$.
It is multiplicatively generated by the fifth degree monomials $e_t(x)$.

More precisely, R^Q consists of elements of degree $0, 5, 10$ and 15, dimensions of the corresponding subspaces are $1, 101, 101$ and 1.
This degree grading defines a Hodge structure on R^Q.

$$\dim R^Q = \dim H_3(X)$$

R^Q is isomorphic to $H^3(X)$ and the isomorphism sends the degree filtration to the Hodge filtration on $H^3(X)$ (Candelas).

Phase symmtery and pairing

There is the greater symmetry group $\mathbb{Z}_5^5$ that diagonally acts on $\mathbb{C}^5$: $\alpha \cdot (x_1, \cdots, x_5) = (\alpha_1 x_1, \cdots, \alpha_5 x_5),\ \alpha_i^5 = 1$.
This action preserves $W_0 = \sum_i x_i^5$.
The above quantum symmetry Q is a subgroup of $\mathbb{Z}_5^5$.

Basis $\{e_\mu(x)\}$ of R^Q is the eigenbasis of the phase symmetry $\mathbb{Z}_5^5$ and each $e_\mu(x)$ has a unique weight.
The phase symmetry preserves the Hodge decomposition.
On the invariant ring R^Q there exists the pairing turning it into a Frobenius algebra:

$$\eta_{\mu\nu} = \mathrm{Res}\, \frac{e_\mu(x)\, e_\nu(x)}{\prod_i \partial_i W_0(x)}.$$

For the monomial basis it is $\eta_{\mu\nu} = \delta_{\mu+\nu,\rho}$.

Two cohomology groups

We introduce a couple of differentials which acts on differential forms on $\mathbb{C}^5$: $D_\pm = \mathrm{d} \mp \mathrm{d}W_0(x)\wedge$.
They define the cohomology groups $H^*_{D_\pm}(\mathbb{C}^5)$.
The isomorphism J between R_0 and $H^*_{D_\pm}(\mathbb{C}^5)$ is defined as

$$J(e_\mu(x)) = e_\mu(x)\,\mathrm{d}^5x,\ e_\mu(x) \in R_0.$$

We see, that $Q = \mathbb{Z}_5$ naturally acts on $H^5_{D_\pm}(\mathbb{C}^5)$ and J sends the Q-invariant part R^Q to Q-invariant subspace $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$.

This space obtains the Hodge structure that corresponds to the Hodge structure on $H^3(X)$.

The complex conjugation acts on $H^3(X)$ so that $\overline{H^{p,q}(X)} = H^{q,p}(X)$, in particular $\overline{H^{2,1}(X)} = H^{1,2}(X)$.

Due to the isomorphism between R^Q and $H^3(X)$ the complex conjugation acts also on the elements of the ring R^Q as $*e_\mu(x) = p_\mu e_{\rho-\mu}(x)$, where p_μ is a constant.

Oscillatory representation

Relative homology groups $H_5(\mathbb{C}^5,\ \mathrm{Re}W_0 = L \to \pm\infty)$ have a natural pairing with Q-invariant cohomology groups $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$:

$$\langle \Gamma^\pm, e_\mu(x)\mathrm{d}^5x\rangle = \int_{\Gamma^\pm} e_\mu(x)e^{\mp W_0(x)}\mathrm{d}^5x,\ \Gamma^\pm \in H_5(\mathbb{C}^5,\ \mathrm{Re}W_0 = L \to \pm\infty$$

Using this we define two invariant homology groups $\mathcal{H}_5^{\pm,inv}$ as quotient of $H_5(\mathbb{C}^5,\ \mathrm{Re}W_0 = L \to \pm\infty)$ by the subgroups orthogonal to $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$.

We introduce the special basises $\Gamma^\pm_\mu$ in the homology groups $\mathcal{H}_5^{\pm,inv}$ by requing their duality to the basises in $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$:

$$\int_{\Gamma^\pm_\mu} e_\nu(x)e^{\mp W_0(x)}\mathrm{d}^5x = \delta_{\mu\nu}$$

and the corresponding periods

$$\sigma^\pm_{\alpha\mu}(\phi) := \int_{\Gamma^\pm_\mu} e_\alpha(x)e^{\mp W(x,\phi)}\mathrm{d}^5x,$$
$$\sigma^\pm_\mu(\phi) := \sigma^\pm_{0\mu}(\phi).$$

Computation of periods $\sigma_\mu(\phi)$

To explicitly compute $\sigma_\mu^\pm(\phi)$, first we expand the exponent in the integral in ϕ representing $W(x,\phi) = W_0(x) + \sum_s \phi_s e_s(x)$

$$\sigma_\mu^\pm(\phi) = \sum_m \int_{\Gamma_\mu^\pm} \prod_r e_r(x)^{m_r}\, e^{\mp W_0(x)}\, \mathrm{d}^5 x \left(\prod_s \frac{(\pm\phi_s)^{m_s}}{m_s!} \right),$$

where $m := \{m_s\}_s,\ m_s \geq 0$ denotes a multi-index of powers of ϕ_s in the expansion above.
$\sigma_\mu^-(\phi) = (-1)^{|\mu|}\sigma_\mu^+(\phi)$, so we focus on $\sigma_\mu(\phi) := \sigma_\mu^+(\phi)$.

For each of the summands the form $\prod_s e_s(x)^{m_s}\, \mathrm{d}^5 x$
belongs to $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$, for it is killed by D_+ and $Q-$invariant.
The oscilatory integrals of D_+-exact terms are zero, therefore:

$$\int_{\Gamma_\mu^+} e^{-W_0(x)} P(x)\, \mathrm{d}^5 x = \int_{\Gamma_\mu^+} e^{-W_0(x)} (P(x)\, \mathrm{d}^5 x + D_+ U)$$

for any polynomial $P(x)$ and any polynomial $4-$form U.

Computation of periods $\sigma_\mu(\phi)$

Let us denote $m_s s_i = 5n_i + \nu_i,\ \nu_i < 5$ for later convenience.
To compute

$$\int_{\Gamma_\mu^+} e^{-W_0(x)} \prod_i x_i^{5n_i+\nu_i}\, \mathrm{d}^5 x$$

we use the trick above with

$$\prod_i x_i^{5n_i+\nu_i}\, \mathrm{d}^5 x =$$

$$= (-1)\left(n_1 - 1 + \frac{\nu_1 + 1}{5}\right) x^{5(n_1-1)+\nu_1} \prod_{i>1} x_i^{5n_i+\nu_i}\, \mathrm{d}^5 x + D_+ U$$

, where

$$U = \frac{1}{5} x_1^{5n_1+\nu_1-4} \prod_{i>1} x_i^{5n_i+\nu_i}\, \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_5$$

Computation of periods $\sigma_\mu(\phi)$

We can continue this procedure by induction with respect to all n_i.
The final result can be compactly written using Pochhammer's symbols:

$$\prod_i x_i^{5n_i+\nu_i}\,\mathrm{d}^5x = (-1)^{\sum_i n_i} \prod_i \left(\frac{\nu_i+1}{5}\right)_{n_i} \prod_i x_i^{\nu_i}\,\mathrm{d}^5x,\ \nu_i < 5.$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$.
If any $\nu_i = 4$ then the differential form is exact and the integral is zero.
Otherwise, rhs of the equation is proportional to $e_\nu(x)$ and we can use the definition of Γ_μ^+:

$$\int_{\Gamma_\mu^+} e_\nu(x)\, e^{-W_0(x)}\,\mathrm{d}^5x = \delta_{\mu\nu}$$

Computation of periods $\sigma_\mu(\phi)$

Doing in this way and integrating over Γ_μ^+ we obtain the explicite expression for the periods

$$\sigma_\mu(\phi) = \sum_{n_i \geq 0} \prod_i \frac{\Gamma\left(n_i + \frac{\mu_i+1}{5}\right)}{\Gamma\left(\frac{\mu_i+1}{5}\right)} \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!}.$$

where

$$\Sigma_n = \{m \mid \sum_s m_s s_i = 5n_i + \mu_i\}$$

Formula for Kähler potential

Pick Lefschetz thimbles $L^{\pm}_{\mu}$ as basis of cycles with real coefficients. Define T as a transition matrix from cycles $\Gamma^{\pm}_{\mu}$ to Lefschetz thimbles $L^{\pm}_{\mu}$

$$\Gamma^{\pm}_{\mu} = (T^{-1})_{\mu\nu}\, L^{\pm}_{\nu}.$$

and compute the transition matrix $T_{\mu\nu}$ with use of the relation

$$T_{\mu\nu} = \int_{L^{\pm}_{\mu}} e_{\nu}(x)\, e^{\mp W_0(x)} \mathrm{d}^5 x.$$

Then we obtain for matrix M as

$$M = T^{-1}\,\bar{T}$$

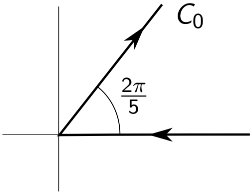
which we need to insert to the expression for Kähler potential together with $\eta_{\mu\nu} = \delta_{\mu,\rho-\nu}$.

Lefschetz thimbles

Lefschetz thimbles $L^{\pm}_{\mu}$ are products of one-dimensional cycles C_{α_i}

$$L^{+}_{\alpha} = \prod_{i=1}^{5} C_{\alpha_i},$$

and $C_{\alpha_i} = \hat{\rho}^{\alpha_i} \cdot C_0$ with $\rho = e^{2\pi i/5}$.
This definition of one-dimensional cycle C_{α_i} means that this cycle is the path in x_i-plane obtained by rotating counter clockwise through angle ρ^{α_i} from the basic path C_0 is depicted on the figure



By construction $L^{\pm}_{\alpha}$ are steepest descent/acsent cycles for $\mathrm{Re}\, W_0$.

Computing the matrices T and M

Now we compute compute $T_{\alpha\mu}$ explicitly

$$T_{\alpha\mu} = \int_{L_\alpha^+} e_\mu \, e^{-W_0} \, \mathrm{d}^5 x = \rho^{(\bar\alpha,\bar\mu)} A(\mu),$$

where A_μ is a product of five Gamma-integrals

$$A_\mu = \frac{1}{5^5} \prod_i \Gamma\left(\frac{\mu_i + 1}{5}\right).$$

Then

$$T^{-1}_{\bar\mu\bar\alpha} = B(\mu)[\bar\rho^{(\bar\mu+1,\bar\alpha)} - 1]$$

$$B(\mu) = \prod_i \frac{1}{\Gamma\left(\frac{\mu_i+1}{5}\right)}$$

$$M_{\mu\nu} = (T^{-1}\bar{T})_{\mu\nu} = \prod_i \gamma\left(\frac{\mu_i + 1}{5}\right) \delta_{\mu,\rho-\nu}$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$$

.

Inserting all these explicit expressions to the formula

$$e^{-K(\phi)} = \sigma^{+}_{\mu}(\phi)\eta_{\mu\lambda}M_{\lambda\nu}\overline{\sigma^{-}_{\nu}(\phi)}$$

we obtain the explicit expression for Kähler potential on the full moduli space of complex structures

$$e^{-K(\phi)} = \sum_{\mu=0}^{203}(-1)^{\deg(\mu)/5}\prod \gamma\left(\frac{\mu_i+1}{5}\right)|\sigma_{\mu}(\phi)|^2,$$

$$\sigma_{\mu}(\phi) = \sum_{n_1,\dots,n_5\geq 0}\prod_{i=1}^{5}\frac{\Gamma(\frac{\mu_i+1}{5}+n_i)}{\Gamma(\frac{\mu_i+1}{5})}\sum_{m\in\Sigma_n}\prod_{s}\frac{\phi_s^{m_s}}{m_s!},$$

$\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$, $0\leq\mu_i\leq 3$, $\quad\sum_{i=1}^{5}\mu_i = 0,5,10,15.$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \qquad \Sigma_n = \{m_s \mid \sum_s m_s s_i = 5n_i+\mu_i\}$$

Appendix A. Coinsidence of two pairings. Proof.

Following Chiodo et al we prove the second equality

$$\eta_{ab} = \mathrm{Res}\, \frac{e_a \cdot e_b \,\mathrm{d}^n x}{\partial_1 W \cdots \partial_n W} = \int_{L_\mu^+} e_a\, e^{-W} \mathrm{d}^n x \; C_{\mu\nu} \int_{L_\nu^-} e_b\, e^{W} \mathrm{d}^n x$$

Consider a small relevant perturbation of the isolated singularity $W(x,t) = W(x) + \sum e_i x_i$. Then the isolated singularity of W located in $x_i = 0$ transforms to a set of Morse critical points $\{p_\mu\}$.
Consider instead of η_{ab} the bilinear form

$$\eta_{ab}(t,z) := \int_{L_\mu^+} e_a\, e^{-W(x,t)/z} \mathrm{d}^n x \; C_{\mu\nu} \int_{L_\nu^-} e_b\, e^{W(x,t)/z} \mathrm{d}^n x$$

First of all we notice, that if $t = 0$, then

$$\eta_{ab}(t=0,z) = z^n \cdot \eta_{ab}(t=0,1).$$

We take as basis of cycles $L_\mu^\pm$ the so-called of Lefschetz thimbles.
They start from Morse points p_μ and goes along the gradient of $\mathrm{Re} W(x,t)$ in the direction of the steepest descent/ascent.
With the proper orientation their intersections are $L_\mu^+ \cap L_\nu^- = \delta_{\mu\nu}$.

Appendix A. Coincidence of two pairings. Proof.

Then rhs of the equality becomes in this basis:

$$\sum_{\mu} \int_{L_\mu^+} e_a \, e^{-W(x,t)/z} \mathrm{d}^n x \, \int_{L_\mu^-} e_b \, e^{W(x,t)/z} \mathrm{d}^n x$$

From stationary phase expansion as $z \to 0$ we obtain for a period:

$$\int_{L_\mu^+} e_a(x) \, e^{-W(x,t)/z} \mathrm{d}^n x = \frac{(2\pi z)^{n/2}}{\sqrt{\mathrm{Hess}\, W(p_\mu, t)}} (e_a(p_\mu) + O(z))$$

Using this we get

$$\eta_{ab}(t,z) = \sum_{\mu} (2\pi i z)^n \frac{e_a(p_\mu) \cdot e_b(p_\mu)}{\mathrm{Hess}(W(p_\mu, t))} (1 + O(z)) =$$

$$(2\pi i z)^n \left(\mathrm{Res} \, \frac{e_a \cdot e_b \, \mathrm{d}^n x}{\partial_1 W \cdots \partial_n W} + O(z) \right)$$

It holds for $t = 0$. Taking into account $\eta_{ab}(0,z) = z^n \cdot \eta_{ab}(0,1)$ we obtain the equality $< \chi_a, \chi_b >= \eta(e_a, e_b)$.

Appendix B. Kähler potential for the moduli space

The final expression for Kähler potential contains periods $\sigma^+_\mu(\phi)$, the pairing $\eta_{\mu\nu}$ and the matrix of the antiholomorphic involution $M_{\mu\nu}$

$$e^{-K(\phi)} = \sum_{\mu,\nu,\lambda} \sigma^+_\mu(\phi)\, \eta_{\mu\lambda}\, M_{\lambda\nu}\, \overline{\sigma^-_\nu(\phi)},$$

Computing periods $\sigma^\pm_\mu(\phi)$ with use the oscilating integrals over $\Gamma^\pm_\mu$ and of the cohomology technique is especially simple.

Basis $\{e_\mu(x)\} \in R^Q$ can be choosen such, that the pairing η is antidiagonal or, equivalently, such that to each $e_\mu(x) {=>} e_{\mu'}(x)$ and $e_\mu(x) \cdot e_{\mu'}(x) = e_\rho(x)$, where $e_\rho(x)$ is the element of degree 3d.

Matrix $M_{\mu\nu} = (T^{-1}\,\bar{T})_{\mu\nu}$ and for this choice is also antidiagonal

$$* e_\mu(x)\, \mathrm{d}^5 x = p_\mu\, e_{\mu'}(x)\, \mathrm{d}^5 x.$$

This simplifies the final expression for K

$$e^{-K(\phi)} = \sum_\mu p_\mu \sigma^+_\mu(\phi)\, \overline{\sigma^-_\mu(\phi)}.$$

Appendix C. Special geometry for Fermat hypersurfaces.

Let X is Fermat CY $X = \{x_1, \ldots, x_5 \ \in \mathbb{P}^4_{(k_1,\ldots,k_5)} | \ W(x,\phi) = 0\}$.

$$W(x,\phi) = \sum_{i=1}^{5} x_i^{\frac{d}{k_i}} + \sum_{\mathbf{s}=\mathbf{1}}^{h_{21}} \phi_{\mathbf{s}} \prod_i x_i^{s_i}, \quad d = \sum_{i=1}^{5} k_i$$

Then Kähler potential on the full Moduli space

$$e^{-K(\phi)} = \sum_{\mu} (-1)^{\deg(\mu)/d} \prod_i \gamma\left(\frac{k_i(\mu_i+1)}{d}\right) |\sigma_\mu(\phi)|^2,$$

$$\sigma_\mu(\phi) = \sum_{n_1,\ldots,n_5 \geq 0} \prod_{i=1}^{5} \frac{\Gamma(\frac{\mu_i+1}{5} + n_i)}{\Gamma(\frac{k_i(\mu_i+1)}{d})} \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!},$$

$0 \leq \mu_i \leq \frac{d}{k_i} - 2, \quad \sum_{i=1}^{5} \mu_i = 0, d, 2d, 3d.$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \qquad \Sigma_n = \{m_{\mathbf{s}} \mid \sum_s m_{\mathbf{s}} s_i = \frac{d}{k_i} n_i + \mu_i\}$$