5D Yang-Mills and modular triple boson

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based on the joint works with Fabrizio Nieri and Yiwen Pan (1710.07170, 1711.06150 and more to come)

Outline

Motivation

supersymmetric localization
(specific method to evaluate path integral)

$$Z = \int D\Phi \; O \; e^{S}$$

such that

$$\delta O = 0 \; , \qquad \delta S = 0 \; ,$$

and $\delta^2 = \mathcal{L}$

$$Z(\tau) = \int D\Phi \; O \; e^{S - \tau \delta W}$$

Independent of τ provided that $\delta^2 W = 0$. So that

$$Z(0) = \lim_{\tau \to \infty} Z(\tau)$$

Motivation

Localization results in various dimensions:

1D-7D gauge theories

compact vs non-compact manifolds

singular configurations

Structural properties of the answer of partition functions/generating functions

5D YM

We take 5D $N = 1$ Yang-Mills theory on:

on $\mathbb{R}^4 \times S^1$ (5D Nekrasov's partition function)

on S^5

we look at their structural properties

5D YM

Nekrasov partition function

$$Z_{\mathbb{R}^4\times S^1}(a, \beta, \epsilon_1, \epsilon_2)$$

$\mathbb{C}^2_{q,t^{-1}} \times S^1$

5D YM

5D YM on S^5

$$Z_{S^5} = \int d^N a \; e^{P(a)} \prod_{i \neq j} S_3(ia_i - ia_j | \vec{\omega}) Z_{\mathbb{R}^4 \times S^1}(a) Z_{\mathbb{R}^4 \times S^1}(a) Z_{\mathbb{R}^4 \times S^1}(a)$$

formal matrix models:

both Nekrasov and S^5 partition functions can be understood as formal matrix models and we can introduce set of times (denoted by t's) and study them as generating functions.

Reminder of Virasoro and q-Virasoro constraints

the Hermitian matrix model:

$$Z(\{t\}) = \int\limits_{u(N)} dM \; e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} \mathrm{Tr}(M^s)} ,$$

where $M^\dagger = M$ and the measure is invariant under $M \to U^\dagger M U$ with $U \in U(N)$.

In terms of eigenvalues of M:

$$Z(\{t\}) = \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \; \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} \sum\limits_{i=1}^{N} \lambda_i^s}$$

Ward identities:

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \sum_{l=1}^{N} \frac{\partial}{\partial \lambda_l} \left(\lambda_l^{n+1} \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} \sum\limits_{i=1}^{N} \lambda_i^s} \right) = 0 \ ,$$

where

$$l_n = - \sum_{l=1}^{N} \frac{\partial}{\partial \lambda_l} \left(\lambda_l^{n+1} \cdots \right)$$

satisfy

$$[l_n, l_m] = (n-m) l_{n+m}$$

Virasoro constraints for Hermitian matrix model

After some rewriting we get the Virasoro constraints:

$$L_n Z(\{t\}) = 0 \ , \quad n \geq -1 \ ,$$

where

$$L_{-1} = \sum_{k=0}^{\infty} t_k \frac{\partial}{\partial t_{k-1}} \ ,$$

$$L_0 = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + N^2 \ ,$$

$$L_n = \sum_{k=0}^{n} (n-k)! k! \frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum_{k=0}^{\infty} \frac{k(k+n)!}{k!} t_k \frac{\partial}{\partial t_{k+n}} \ , \quad n \geq 1$$

Virasoro constraints for Hermitian matrix model

Let us think for the moment, we naturally have the representation of Heisenberg algebra:

$$\text{creation operator:} \quad \alpha_{-n} = \frac{\sqrt{2}}{(n-1)!} t_n \ ,$$

$$\text{annihilation operator:} \quad \alpha_n = \frac{n!}{\sqrt{2}} \frac{\partial}{\partial t_n} \ ,$$

and we can check that

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_{n-m} \alpha_m : \ , \quad n \geq -1$$

but it can be extended to all n's and we get the full Virasoro algebra with the central charge $c = 1$.

Virasoro constraints for Hermitian matrix model

Thus we deal with the free boson $\phi(x) = \sum_n a_n x^{-n}$

Looking for an operator $S(x)$ such that

$$[L_n, S(x)] = \frac{d}{dx} \mathrm{O}(x) \ ,$$

we can get easily the solution of Virasoro constraints

$$Z(\{t\}) = \mathrm{Q}^N \ , \quad \mathrm{Q} = \int dx \ S(x) \ ,$$

we immediately get

$$L_n Q^N |0\rangle = L_n(\{t_k\}) \mathrm{Z}(\{t_k\}) = 0$$

This is indeed the Hermitian matrix model, in this argument only contour of integration is not specified.

Virasoro constraints can be deformed and more measures can be obtained

Deformations of Heisenberg ($p = qt^{-1}$, $t = q^{\beta}$):

$$[a_n, a_m] = \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\delta_{n+m,0} , \quad n, m \in \mathbb{Z}\backslash\{0\} ,$$

$$[P, Q] = 2 ,$$

the deformed Virasoro

$$[T_n, T_m] = -\sum_{\ell} f_{\ell}(T_{n-\ell}T_{m+\ell} - T_{m-\ell}T_{n+\ell})$$
$$-\frac{(1-q)(1-t^{-1})}{(1-p)}(p^n - p^{-n})\delta_{n+m,0}$$

$q = e^{\hbar}$, we have the small $\hbar$ expansion

$$T_n = 2\delta_{n,0} + \hbar^2\beta\left(L_n + \frac{Q_\beta^2}{4}\delta_{n,0}\right) + O(\hbar^4)$$

the representation of deformed Heisenberg

$$a_{-n} = (q^{\frac{n}{2}} - q^{-\frac{n}{2}})t_n\ , \quad a_n = \frac{1}{n}(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\frac{\partial}{\partial t_n}\ , \quad n \in \mathbb{Z}_{>0}\ ,$$

$$\sqrt{\beta}Q = t_0\ , \quad P = 2\sqrt{\beta}\frac{\partial}{\partial t_0}\ , \quad |0>= 1$$

So we do the similar thing, construct the operators S such that

$$[T_n, \int dx\ S(x)] = 0$$

q-Virasoro

$$Z(\{t\}) = \oint \prod_{i=1}^{N} \frac{dw_i}{2\pi i w_i} \prod_{i\neq j} \frac{(w_i w_j^{-1}; q)_\infty}{(t w_i w_j^{-1}; q)_\infty} e^{\sum\limits_{k=0}^{\infty} t_k \sum\limits_j w_j^k}$$

such that

$$T_n Z(\{t\}) = 0 \ , \quad n > 0$$

3D gauge theory on $D^2 \times S^1$,

$N = 2$ $U(N)$ vector with adjoint chiral

Instead of integral we can have sum!

q-Virasoro vs 5D YM

q-Virasoro for $Z_{\mathbb{R}^4\times S^1}(a,\{t\})$

$$T_n Z_{Nekr}(a,\{t\}) = 0\ , \quad n>0$$

q-Virasoro for $Z_{S^5}(\{t\},\{\tilde{t}\},\{\tilde{\tilde{t}}\})$

$$T_n Z_{S^5} = 0\ , \quad n>0$$

$$\tilde{T}_n Z_{S^5} = 0\ , \quad n>0$$

$$\tilde{\tilde{T}}_n Z_{S^5} = 0\ , \quad n>0$$

q-Virasoro vs 5D YM

$$\left(q_1 = {}^{2\pi i\tau},\, t_1 = {}^{2\pi i\sigma}\right)$$

$$\left(q_2 = {}^{-2\pi i\sigma/\tau},\, t_2 = {}^{-2\pi i\tau}\right) \longmapsto \left(q_3 = {}^{-2\pi i\sigma},\, t_3 = {}^{2\pi i\tau/\sigma}\right) .$$

$\tau = \omega_2/\omega_1$ and $\sigma = \omega_3/\omega_1$

$$\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 + \omega_3^2|z_3|^2 = 1$$

5D/3D/1D

$$Z_{\mathbb{C}^2_{q,t^{-1}}\times S^1} = \sum_{r,c\geq 0} \int Z_{\mathbb{C}^1_q}(u(r)) Z_{\mathbb{C}^1_{t^{-1}}}(u(c)) Z_{S^1}$$

q-Virasoro perspective and direct calculation

5D/3D/1D

$$Z_{S^5} = \sum_{N_1, N_2, N_3 \geq 0} Z_{S^3}(N_1) Z_{S^3}(N_2) Z_{S^3}(N_3) Z_{S^1} Z_{S^1} Z_{S^1}$$

q-Virasoro perspective and direct calculation

5D/3D/1D

three copies of (deformed) Heisenberg algebra: $a_n, \tilde{a}_n, \tilde{\tilde{a}}_n$

construct screening charge commuting with all three q-Virasoro (modular triple)

$$\phi(x) = \sum_n a_n e^{2\pi i n \frac{x}{\omega_1}} + \tilde{a}_n e^{2\pi i n \frac{x}{\omega_2}} + \tilde{\tilde{a}}_n e^{2\pi i n \frac{x}{\omega_3}}$$

boson

we can construct the formal boson theory

$$S = \int dx \; \partial_x d_{\omega_i} \phi \; d_{\omega_j} d_{\omega_k} \phi$$

where

$$d_\omega \phi(x) = \phi(x + \omega/2) - \phi(x - \omega/2)$$

boson

Green function

$$\partial_x d_{\omega_1} d_{\omega_2} d_{\omega_3} \log S_3(x|\vec{\omega}) = \delta(x)$$

The system has 3 copies of Heisenberg algebra and related to q-Virasoro construction of S^5 partition function

$$\int dx\ e^{\alpha x} e^{(\omega, d_{\omega_i} \phi)}$$

Summary

- 5D answer can be written in 3D and 1D terms
- what does it mean from the point of view of localization
- many puzzling structural properties