Representations of quantum groups and functional relations in integrable systems

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- **The birth of a new science: QISM**

  Faddeev's School of Mathematics, 1970's – 1980's

- **Quantum groups**

  Drinfeld, Jimbo, 1985 – 1986

- **Quantum group approach to functional relations**

  Bazhanov, Lukyanov, Zamolodchikov, 1994 – 2001

- **Group-theoretic / Algebraic approach revisited**

  Boos, Göhmann, Klümper, NX, Razumov, 2010 – 2017

- **Prefundamental representations and functional relations**

  Hernandez, Jimbo, 2011
  Frenkel, Hernandez, 2014

Quantum groups: V. Drinfeld & M. Jimbo (1985 - 1986)

- $\mathcal{A}$ is a Hopf algebra with respect to Δ, S, ε
- $\mathcal{A}$ is a Hopf algebra with $\Delta^{\mathrm{op}} = \Pi \circ \Delta$

$$\Pi(a \otimes b) = b \otimes a, \qquad a, b \in \mathcal{A}$$

- The universal R-matrix

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\,\Delta(a)\mathcal{R}^{-1}, \qquad \mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_- \subset \mathcal{A} \otimes \mathcal{A}$$

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{13}\,\mathcal{R}^{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\,\mathcal{R}^{12}$$

- The master equation

$$\mathcal{R}^{12}\,\mathcal{R}^{13}\,\mathcal{R}^{23} = \mathcal{R}^{23}\,\mathcal{R}^{13}\,\mathcal{R}^{12}$$

defined in $\mathcal{B}_+ \otimes \mathcal{A} \otimes \mathcal{B}_- \subset \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

Universal integrability objects

Introduce a twist element t: $\quad \Delta(t) = t \otimes t$

- Monodromy operator: $\quad \varphi : \mathcal{A} \to \mathrm{End}(V)$

$$\mathcal{M}_\varphi = (\varphi \otimes \mathrm{id})(\mathcal{R}) \in \mathrm{End}(V) \otimes \mathcal{B}_-$$

- Transfer operator

$$\mathcal{T}_\varphi = (\mathrm{tr}_V \otimes \mathrm{id})(\mathcal{M}_\varphi(\varphi(t) \otimes 1)) = ((\mathrm{tr}_V \circ \varphi) \otimes \mathrm{id})(\mathcal{R}(t \otimes 1))$$

- *L*-operator: $\quad \rho : \mathcal{B}_+ \to \mathrm{End}(W)$

$$\mathcal{L}_\rho = (\rho \otimes \mathrm{id})(\mathcal{R}) \in \mathrm{End}(W) \otimes \mathcal{B}_-$$

- *Q*-operator

$$\mathcal{Q}_\rho = (\mathrm{tr}_W \otimes \mathrm{id})(\mathcal{L}_\rho(\rho(t) \otimes 1)) = ((\mathrm{tr}_W \circ \rho) \otimes \mathrm{id})(\mathcal{R}(t \otimes 1))$$

First universal functional relations

- Modified Yang–Baxter equation

$$(\mathcal{R}^{13}t^1)(\mathcal{R}^{23}t^2) = (\mathcal{R}^{12})^{-1}(\mathcal{R}^{23}t^2)(\mathcal{R}^{13}t^1)\mathcal{R}^{12} \qquad (*)$$

- Directly from the Yang-Baxter equation

$$((\mathrm{tr} \circ \varphi_1) \otimes (\mathrm{tr} \circ \varphi_2))(*) \Rightarrow \mathcal{T}_{\varphi_1}\mathcal{T}_{\varphi_2} = \mathcal{T}_{\varphi_2}\mathcal{T}_{\varphi_1}$$

$$((\mathrm{tr} \circ \rho) \otimes (\mathrm{tr} \circ \varphi))(*) \Rightarrow \mathcal{Q}_\rho\mathcal{T}_\varphi = \mathcal{T}_\varphi\mathcal{Q}_\rho$$

- The rest is more tricky:

$$\mathcal{Q}_{\rho_1}\mathcal{Q}_{\rho_2} = ((\mathrm{tr}_{W_1 \otimes W_2} \circ (\rho_1 \otimes \rho_2)) \otimes \mathrm{id})\left(\mathcal{R}^{13}t^1\mathcal{R}^{23}t^2\right)$$

$$\mathcal{R}^{13}t^1\mathcal{R}^{23}t^2 = [(\Delta \otimes \mathrm{id})(\mathcal{R})]\,[(\Delta \otimes \mathrm{id})(t \otimes 1)] = (\Delta \otimes \mathrm{id})(\mathcal{R}(t \otimes 1))$$

$$\mathcal{Q}_{\rho_1}\mathcal{Q}_{\rho_2} = ((\mathrm{tr}_{W_1 \otimes W_2} \circ (\rho_1 \otimes_\Delta \rho_2)) \otimes \mathrm{id})(\mathcal{R}(t \otimes 1))$$

Simplest example

- Monodromy matrix

$$\mathbb{M}(\zeta) = C_2^{1/4}\, e^{F_2(\zeta^s)} \begin{pmatrix} q^{-K_1} - q^{K_1}\,\zeta^s & \kappa_q q^{K_1}\, F\, \zeta^{s_0} \\ \kappa_q E\, q^{-K_1}\,\zeta^{s_1} & q^{-K_2} - q^{K_2}\,\zeta^s \end{pmatrix}$$

$$C_2 = q^{2(K_1+K_2)}, \qquad F_2(\zeta) = \sum_{n\in\mathbb{N}} \frac{1}{q^n + q^{-n}} F_n \frac{\zeta^n}{n}, \qquad F_n \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{gl}_2))$$

- Transfer matrix $\widetilde{\mathbb{T}}^\lambda(\zeta) = \widetilde{\mathrm{tr}}^\lambda(\mathbb{M}(\zeta)\, q^{\phi H})$

$$\widetilde{\mathbb{T}}^\lambda(\zeta) = q^{(\lambda_1+\lambda_2)/2} e^{\Lambda_2(\zeta^s)} \begin{pmatrix} \frac{q^{-\lambda_1}}{1-q^{1-2\phi}} - \frac{q^{\lambda_1}\,\zeta^s}{1-q^{-1-2\phi}} & 0 \\ 0 & \frac{q^{-\lambda_2}}{1-q^{-1-2\phi}} - \frac{q^{\lambda_2}\,\zeta^s}{1-q^{1-2\phi}} \end{pmatrix}$$

$$\Lambda_2(\zeta) = \sum_{n\in\mathbb{N}} \frac{q^{(2\lambda_1+1)n} + q^{(2\lambda_2-1)n}}{q^n + q^{-n}} \frac{\zeta^n}{n}$$

Simplest example

- L-operators

$$\mathbb{L}(\zeta) = e^{f_2(\zeta^s)} \begin{pmatrix} q^{-N} - q^{N+1}\zeta^s & bq^{2N}\zeta^{s_0} \\ -\kappa_q b^\dagger q^{-N}\zeta^{s_1} & q^N \end{pmatrix}$$

$$\bar{\mathbb{L}}(\zeta) = e^{f_2(\zeta^s)} \begin{pmatrix} q^N & -\kappa_q b^\dagger q^{-N}\zeta^{s_1} \\ bq^{2N}\zeta^{s_0} & q^{-N} - q^{N+1}\zeta^s \end{pmatrix}$$

$$f_2(\zeta) = \sum_{n\in\mathbb{N}} \frac{1}{q^n + q^{-n}} \frac{\zeta^n}{n}$$

- Q-operators

$$\mathbb{Q}(\zeta) = \mathrm{tr}(\mathbb{L}(\zeta) q^{-2\phi N}) = e^{f_2(\zeta^s)} \begin{pmatrix} \frac{1}{1-q^{-1+2\phi}} - \frac{q\,\zeta^s}{1-q^{1+2\phi}} & 0 \\ 0 & \frac{1}{1-q^{1+2\phi}} \end{pmatrix}$$

$$\bar{\mathbb{Q}}(\zeta) = \mathrm{tr}(\bar{\mathbb{L}}(\zeta) q^{2\phi N}) = e^{f_2(\zeta^s)} \begin{pmatrix} \frac{1}{1-q^{1-2\phi}} & 0 \\ 0 & \frac{1}{1-q^{-1-2\phi}} - \frac{q\,\zeta^s}{1-q^{1-2\phi}} \end{pmatrix}$$

- Observing functional relation

$$\mathbb{D}^\lambda\, \widetilde{\mathbb{T}}^\lambda(\zeta) = \mathbb{Q}(q^{(2\lambda_1+1)/s}\zeta)\, \bar{\mathbb{Q}}(q^{(2\lambda_2-1)/s}\zeta)$$

Quantum group $\mathrm{U}_q(\mathfrak{gl}_{l+1})$

- Generators of $\mathrm{U}_q(\mathfrak{gl}_{l+1})$

$$E_i, \quad F_i, \quad i = 1, \dots, l, \qquad q^X, \quad X \in \mathfrak{k}_{l+1} = \bigoplus_{i=1}^{l+1} \mathbb{C} K_i$$

- Dual basis and simple roots

$$\epsilon_i \in \mathfrak{k}^*_{l+1}, \quad \langle \epsilon_i, K_j \rangle = \delta_{ij}, \qquad \alpha_i \in \mathfrak{k}^*_{l+1}, \quad \alpha_i = \epsilon_i - \epsilon_{i+1}$$

- Positive roots

$$\alpha_{ij} = \epsilon_i - \epsilon_j = \sum_{k=i}^{j-1} \alpha_k, \qquad \Lambda_l = \{i, j \in \mathbb{N} \times \mathbb{N} \,|\, 1 \le i < j \le l + 1\}$$

- Root vectors

$$E_{i,i+1} = E_i, \qquad F_{i,i+1} = F_i, \qquad i = 1, \dots, l$$

$$E_{ij} = E_{ik} E_{kj} - q E_{kj} E_{ik}, \qquad F_{ij} = F_{kj} F_{ik} - q^{-1} F_{ik} F_{kj}, \qquad i < k < j$$

Poincaré–Birkhoff–Witt basis and Verma $\mathrm{U}_q(\mathfrak{gl}_{l+1})$-modules

- Monomials

$$\{F_{i_1j_1}\dots F_{i_aj_a}\, q^{\nu_1K_1}\dots q^{\nu_cK_c}\, E_{m_1n_1}\dots E_{m_bn_b} \mid a,b,c \geq 0\}$$

  form a basis of $\mathrm{U}_q(\mathfrak{gl}_{l+1})$, where

$$(i_1j_1) \preceq \dots \preceq (i_aj_a), \qquad (m_1n_1) \preceq \dots \preceq (m_bn_b)$$

  imply a normal order on the set $\{\alpha_{ij} \mid (ij) \in \Lambda_l\}$

- Given a highest weight vector $v^\lambda = v_{\mathbf{0}}$

$$E_i v_{\mathbf{0}} = 0, \quad i = 1,\dots,l, \qquad q^X v_{\mathbf{0}} = q^{\langle\lambda,X\rangle} v_{\mathbf{0}}, \quad X \in \mathfrak{k}_{l+1}, \quad \lambda \in \mathfrak{k}^*_{l+1}$$

  the vectors

$$v_{\boldsymbol{m}} = \prod_{(ij)=(12)}^{(l,l+1)} F_{ij}^{m_{ij}}\, v_{\mathbf{0}}, \qquad \boldsymbol{m} = (m_{12},\dots,m_{l,l+1}) \in \mathbb{Z}_+^{\otimes\, l(l+1)/2}$$

  form a basis of $\mathrm{U}_q(\mathfrak{gl}_{l+1})$-module $\widetilde{V}^\lambda$

- Finite-dimensional module V^λ is a quotient of $\widetilde{V}^\lambda$ over the maximal submodule if $\langle\lambda, K_i - K_{i+1}\rangle \in \mathbb{Z}_+$ for all $i = 1,\dots,l$

Quantum loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

- Lie algebra $\widetilde{\mathcal{L}}(\mathfrak{sl}_{l+1})$: generators, Cartan subalgebra and center

$$e_i, \quad f_i, \quad h_i, \quad i = 0, 1, \ldots, l, \qquad \widetilde{\mathfrak{h}}_{l+1} = \bigoplus_{i=0}^{l} \mathbb{C} h_i, \qquad c = \sum_{i=0}^{l} h_i$$

- Quantum group $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_{l+1}))$ generated by

$$e_i, \quad f_i, \quad i = 0, 1, \ldots, l, \qquad q^x, \quad x \in \widetilde{\mathfrak{h}}_{l+1}$$

- $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_{l+1}))$ has no finite-dimensional representation with $q^{\nu c} \neq 1$, *hence*

$$\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1})) = \mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_{l+1})) / \langle q^{\nu c} - 1 \rangle_{\nu \in \mathbb{C}}$$

- The monomials

$$\{ f_{\gamma_1}^{m_1} \cdots f_{\gamma_a}^{m_a} \, q^x \, e_{\delta_1}^{n_1} \cdots e_{\delta_b}^{n_b} \quad | \quad m_i, n_i \geq 0 \}$$

  with a normal order of the roots $\gamma_1 \preceq \ldots \preceq \gamma_a, \quad \delta_1 \preceq \ldots \preceq \delta_b$, form a Poincaré–Birkhoff–Witt basis of the quantum loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

Jimbo's homomorphism and representations of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

- Basic representation $\boxed{\widetilde{\varphi}^{\lambda}_{\zeta} = \widetilde{\pi}^{\lambda} \circ \varepsilon \circ \mathit{\Gamma}_{\zeta}}$

$$\mathit{\Gamma}_{\zeta}(q^{x}) = q^{x}, \qquad \mathit{\Gamma}_{\zeta}(e_i) = \zeta^{s_i} e_i, \qquad \mathit{\Gamma}_{\zeta}(f_i) = \zeta^{-s_i} f_i$$

$$\underline{\varepsilon : \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1})) \to \mathrm{U}_q(\mathfrak{gl}_{l+1})}$$

$$\varepsilon(q^{\nu h_0}) = q^{\nu(K_{l+1} - K_1)}, \qquad \varepsilon(q^{\nu h_i}) = q^{\nu(K_i - K_{i+1})}$$

$$\varepsilon(e_0) = F_{1,l+1}\, q^{K_1 + K_{l+1}}, \qquad \varepsilon(e_i) = E_{i,i+1}$$

$$\varepsilon(f_0) = E_{1,l+1}\, q^{-K_1 - K_{l+1}}, \qquad \varepsilon(f_i) = F_{i,i+1}$$

- $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$-module relations

$$\{q^{\nu h_0}\, v_{\boldsymbol{m}}, \quad q^{\nu h_i}\, v_{\boldsymbol{m}}, \quad e_0\, v_{\boldsymbol{m}}, \quad e_i\, v_{\boldsymbol{m}}, \quad f_0\, v_{\boldsymbol{m}}, \quad f_i\, v_{\boldsymbol{m}} \mid i = 1, \ldots, l\}$$

More universal integrability objects

- Automorphisms σ and τ

$$\sigma(A_l^{(1)}) \ : \ \alpha_i \to \alpha_{i+1}, \qquad i = 0, 1, \ldots, l, \qquad \sigma^{l+1} = \mathrm{id}$$
$$\tau(A_l^{(1)}) \ : \ \alpha_0 \to \alpha_0, \qquad \alpha_i \to \alpha_{l+1-i}, \qquad i = 1, \ldots, l, \qquad \tau^2 = \mathrm{id}$$

- And more homomorphisms

$$\rho : \mathrm{U}_q(\mathfrak{b}^+) \to W, \qquad \rho_\zeta = \rho \circ \Gamma_\zeta, \qquad \theta_\zeta = \chi \circ \rho_\zeta$$
$$\rho_a = \rho \circ \sigma^{-a}, \qquad \bar{\rho}_a = \rho \circ \tau \circ \sigma^{-a+1}$$

- Universal L-operators

$$\mathcal{L}_a(\zeta) = (\rho_{a\zeta} \otimes \mathrm{id})(\mathcal{R}), \qquad \bar{\mathcal{L}}_a(\zeta) = (\bar{\rho}_{a\zeta} \otimes \mathrm{id})(\mathcal{R})$$

- Universal Q-operators

$$\mathcal{Q}_a(\zeta) = ((\mathrm{tr} \circ \theta_{a\zeta}) \otimes \mathrm{id})(\mathcal{R}(t \otimes 1)), \qquad \bar{\mathcal{Q}}_a(\zeta) = ((\mathrm{tr} \circ \bar{\theta}_{a\zeta}) \otimes \mathrm{id})(\mathcal{R}(t \otimes 1))$$

Main functional relations

- Cipher key: the product representation

$$\mathcal{C}\,\widetilde{\mathcal{T}}^{\lambda}(\zeta) = \mathcal{Q}_1(q^{2(\lambda+\rho)_1/s}\zeta)\cdots\mathcal{Q}_{l+1}(q^{2(\lambda+\rho)_{l+1}/s}\zeta)$$

The element $\rho \in \mathfrak{k}^*_{l+1}$ is the half-sum of the positive roots

$$\rho = (l/2,\, (l-2)/2,\, \ldots,\, -l/2)$$

- The quantum B. G. G. resolution connects the traces

$$\mathrm{tr}^{\lambda} = \sum_{w\in\mathfrak{W}} (-1)^{\ell(w)}\,\widetilde{\mathrm{tr}}^{\,w\cdot\lambda}, \qquad \mathcal{T}^{\lambda}(\zeta) = \sum_{p\in \mathrm{S}_{l+1}} \mathrm{sgn}(p)\,\widetilde{\mathcal{T}}^{\,p(\lambda+\rho)-\rho}(\zeta)$$

and leads to the determinant representation $\quad (i, j = 1, \ldots, l+1)$

$$\mathcal{C}\,\mathcal{T}^{\lambda-\rho}(\zeta) = \det\left(\mathcal{Q}_i(q^{2\lambda_j/s}\zeta)\right), \qquad \mathcal{C}\,\overline{\mathcal{T}}^{\lambda-\rho}(\zeta) = \det\left(\overline{\mathcal{Q}}_i(q^{-2\lambda_j/s}\zeta)\right)$$

Simplest higher-rank example

$$\boxed{l=2}$$

- *TQ*-relations

$$\mathcal{T}^{(1,1,0)}(\zeta)\mathcal{Q}_j(\zeta)-\mathcal{T}^{(1,0,0)}(\zeta)\mathcal{Q}_j(q^{-2/s}\zeta)=\mathcal{Q}_j(q^{2/s}\zeta)-\mathcal{Q}_j(q^{-4/s}\zeta)$$

$$\bar{\mathcal{T}}^{(1,1,0)}(\zeta)\bar{\mathcal{Q}}_j(\zeta)-\bar{\mathcal{T}}^{(1,0,0)}(\zeta)\bar{\mathcal{Q}}_j(q^{2/s}\zeta)=\bar{\mathcal{Q}}_j(q^{-2/s}\zeta)-\bar{\mathcal{Q}}_j(q^{4/s}\zeta)$$

- *TT*-relations

$$\mathcal{T}^{(k-1,0,0)}(\zeta)\mathcal{T}^{(k+1,0,0)}(q^{2/s}\zeta)=\mathcal{T}^{(k,0,0)}(\zeta)\mathcal{T}^{(k,0,0)}(q^{2/s}\zeta)-\mathcal{T}^{(k,k,0)}(\zeta)$$

$$\mathcal{T}^{(k-1,k-1,0)}(q^{-2/s}\zeta)\mathcal{T}^{(k+1,k+1,0)}(\zeta)=\mathcal{T}^{(k,k,0)}(q^{-2/s}\zeta)\mathcal{T}^{(k,k,0)}(\zeta)-\mathcal{T}^{(k,0,0)}(\zeta)$$

Drinfeld–Jimbo's versus 2nd Drinfeld's

- Generators $\xi^{\pm}_{i,n}$ and $\chi_{i,n}$

$$\xi^+_{i,n} = \begin{cases} (-1)^{ni} e_{\alpha_i + n\delta} & n \geq 0 \\ -(-1)^{ni} q_i^{-h_i} f_{(\delta-\alpha_i)-(n+1)\delta} & n < 0 \end{cases}$$

$$\xi^-_{i,n} = \begin{cases} -(-1)^{(n+1)i} e_{(\delta-\alpha_i)+(n-1)\delta}\, q_i^{h_i} & n > 0 \\ (-1)^{ni} f_{\alpha_i - n\delta} & n \leq 0 \end{cases}$$

$$\chi_{i,n} = \begin{cases} -(-1)^{ni} e_{n\delta,\alpha_i} & n > 0 \\ -(-1)^{ni} f_{-n\delta,\alpha_i} & n < 0 \end{cases}$$

- Generators $\phi^{\pm}_{i,n}$

$$\phi^+_{i,n} = \begin{cases} -(-1)^{ni} \kappa_q\, q_i^{h_i} e'_{n\delta,\alpha_i} & n > 0 \\ q_i^{h_i} & n = 0 \end{cases}$$

$$\phi^-_{i,n} = \begin{cases} q_i^{-h_i} & n = 0 \\ (-1)^{ni} \kappa_q\, q_i^{-h_i} f'_{-n\delta,\alpha_i} & n < 0 \end{cases}$$

$$\boxed{\phi_i^+(u) = q_i^{h_i}\big(1 - \kappa_q\, e'_{\delta,\alpha_i}((-1)^i u)\big), \quad \phi_i^-(u^{-1}) = q_i^{-h_i}\big(1 + \kappa_q f'_{\delta,\alpha_i}((-1)^i u^{-1})\big)}$$

Highest ℓ-weight $\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$-modules

- Highest ℓ-weight $\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$-module V (*in the category* $\mathcal{O}$) with highest ℓ-weight $\quad \boldsymbol{\Psi} = \big(\lambda, \boldsymbol{\Psi}^+, \boldsymbol{\Psi}^-\big)$

$$\lambda \in \mathfrak{h}^*_{l+1}, \quad \boldsymbol{\Psi}^+ = (\Psi_i^+(u))_{i \in I} \in \mathbb{C}[[u]], \quad \boldsymbol{\Psi}^- = (\Psi_i^-(u))_{i \in I} \in \mathbb{C}[[u^{-1}]]$$

$$\exists\, v \in V : \quad \phi_i^+(u)\, v = \Psi_i^+(u)\, v, \quad \phi_i^-(u^{-1})\, v = \Psi_i^-(u^{-1})\, v, \quad i \in I$$

$$\xi_{i,n}^+ v = 0, \qquad i \in I,\ n \in \mathbb{Z}, \qquad V = \mathrm{U}_q(\mathcal{L}(\mathfrak{g}))\, v$$

- Rational ℓ-weights

$$\Psi_i^+(u) = \frac{a_{ip_i} u^{p_i} + a_{i,p_i-1} u^{p_i-1} + \cdots + a_{i0}}{b_{ip_i} u^{p_i} + b_{i,p_i-1} u^{p_i-1} + \cdots + b_{i0}}$$

$$\Psi_i^-(u^{-1}) = \frac{a_{ip_i} + a_{i,p_i-1} u^{-1} + \cdots + a_{i0} u^{-p_i}}{b_{ip_i} + b_{i,p_i-1} u^{-1} + \cdots + b_{i0} u^{-p_i}}$$

$$\frac{a_{i0}}{b_{i0}} = q^{\langle \lambda, h_i \rangle}, \qquad \frac{a_{ip_i}}{b_{ip_i}} = q^{-\langle \lambda, h_i \rangle}$$

Highest ℓ-weight $\mathrm{U}_q(\mathfrak{b}^+)$-modules

- Highest ℓ-weight $\mathrm{U}_q(\mathfrak{b}^+)$-module W (*in the category* $\mathcal{O}$) with highest ℓ-weight $\quad \boldsymbol{\Psi} = \big(\lambda, \boldsymbol{\Psi}^+\big)$

$$\exists\, v \in W : \quad \phi_i^+(u)\, v = \Psi_i^+(u)\, v, \qquad i \in I$$

$$\xi_{i,n}^+ v = 0, \qquad i \in I,\ n \in \mathbb{Z}_+, \qquad W = \mathrm{U}_q(\mathfrak{b}^+)\, v$$

- Rational ℓ-weights

$$\Psi_i^+(u) = \frac{a_{ip_i} u^{p_i} + a_{i,p_i-1} u^{p_i-1} + \cdots + a_{i0}}{b_{iq_i} u^{q_i} + b_{i,q_i-1} u^{q_i-1} + \cdots + b_{i0}}$$

$$\frac{a_{i0}}{b_{i0}} = q^{\langle \lambda, h_i \rangle}$$

- Prefundamental representations

$$L_{i,z}^{\pm} : \qquad \boldsymbol{\Psi}^+ = (\underbrace{1, \ldots, 1}_{i-1}, (1 - zu)^{\pm 1}, \underbrace{1, \ldots, 1}_{l-i}), \qquad i \in I, \quad z \in \mathbb{C}^\times$$

$$L_\xi : \qquad \lambda_\xi = \xi \in \mathfrak{h}_{l+1}^*, \qquad (\boldsymbol{\Psi}_\xi)^+ = (q^{\langle \xi, h_1 \rangle}, \ldots, q^{\langle \xi, h_l \rangle})$$

Basic properties of highest ℓ-weight modules

- Product of ℓ-weights

$$\boldsymbol{\Psi}_1 = \left(\lambda_1, \boldsymbol{\Psi}_1^+, \boldsymbol{\Psi}_1^-\right), \qquad \boldsymbol{\Psi}_1 = \left(\lambda_2, \boldsymbol{\Psi}_2^+, \boldsymbol{\Psi}_2^-\right)$$

$$\boldsymbol{\Psi}_1 \boldsymbol{\Psi}_2 = \left(\lambda_1 + \lambda_2,\, \boldsymbol{\Psi}_1^+ \boldsymbol{\Psi}_2^+,\, \boldsymbol{\Psi}_1^- \boldsymbol{\Psi}_2^-\right)$$

$$\boldsymbol{\Psi}_1^+ \boldsymbol{\Psi}_2^+ = (\Psi_{1i}^+(u) \Psi_{2i}^+(u))_{i \in I}, \quad \boldsymbol{\Psi}_1^- \boldsymbol{\Psi}_2^- = (\Psi_{1i}^-(u^{-1}) \Psi_{2i}^-(u^{-1}))_{i \in I}$$

- Submodules of $\quad L(\boldsymbol{\Psi}_1) \otimes L(\boldsymbol{\Psi}_2)$

$$\boldsymbol{\Psi}_1,\, \boldsymbol{\Psi}_2 : \quad L(\boldsymbol{\Psi}_1 \boldsymbol{\Psi}_2) \cong L(\boldsymbol{\Psi}_1)\, \overline{\otimes}\, L(\boldsymbol{\Psi}_2)$$

- $\{$ Rational ℓ-weights $\boldsymbol{\Psi}$ $\} \leftrightarrow \{[$ Simple $\mathrm{U}_q(\mathcal{L}(\mathfrak{g}))$-modules $L(\boldsymbol{\Psi})$ in $\mathcal{O}$ $]\}$

- $\{$ Rational ℓ-weights $\boldsymbol{\Psi}$ $\} \leftrightarrow \{[$ Simple $\mathrm{U}_q(\mathfrak{b}^+)$-modules $L(\boldsymbol{\Psi})$ in $\mathcal{O}$ $]\}$

Homomorphisms ρ_a, $\bar{\rho}_a$ & representations θ_a, $\bar{\theta}_a$

- Homomorphisms ρ_a, representations θ_a, bases $v_{\mathbf{m}}^{(a)}$

$$\rho_a = \rho \circ \sigma^{-a}, \qquad a = 1, \ldots, l+1$$

$$\theta_a = (\underbrace{\chi^- \otimes \cdots \otimes \chi^-}_{l-a+1} \otimes \underbrace{\chi^+ \otimes \cdots \otimes \chi^+}_{a-1}) \circ \rho_a$$

$$v_{\mathbf{m}}^{(a)} = b_1^{m_1} \cdots b_{l-a+1}^{m_{l-a+1}} \, b_{l-a+2}^{\dagger\, m_{l-a+2}} \cdots b_l^{\dagger\, m_l} \, v_{\mathbf{0}}$$

- Homomorphisms $\bar{\rho}_a$, representations $\bar{\theta}_a$, bases $\bar{v}_{\mathbf{m}}^{(a)}$

$$\bar{\rho}_a = \rho \circ \tau \circ \sigma^{-a+1}, \qquad a = 1, \ldots, l+1$$

$$\bar{\theta}_a = (\underbrace{\chi^- \otimes \cdots \otimes \chi^-}_{a-1} \otimes \underbrace{\chi^+ \otimes \cdots \otimes \chi^+}_{l-a+1}) \circ \bar{\rho}_a$$

$$\bar{v}_{\mathbf{m}}^{(a)} = b_1^{m_1} \cdots b_{a-1}^{m_{a-1}} \, b_a^{\dagger\, m_a} \cdots b_l^{\dagger\, m_l} \, v_{\mathbf{0}}$$

Calculating the ℓ-weights

- We define

$$(\theta_a)_\zeta = \theta_a \circ \Gamma_\zeta, \qquad (\bar\theta_a)_\zeta = \bar\theta_a \circ \Gamma_\zeta$$

  and put

$$(\theta_a)_\zeta(\phi_i^+(u))\, v_{\boldsymbol{m}}^{(a)} = \theta_a(\phi_i^+(\zeta^s u))\, v_{\boldsymbol{m}}^{(a)} = \Psi^+_{i,\boldsymbol{m},a}(u)\, v_{\boldsymbol{m}}^{(a)}$$

$$(\bar\theta_a)_\zeta(\phi_i^+(u))\, \bar v_{\boldsymbol{m}}^{(a)} = \bar\theta_a(\phi_i^+(\zeta^s u))\, \bar v_{\boldsymbol{m}}^{(a)} = \overline{\Psi}^+_{i,\boldsymbol{m},a}(u)\, \bar v_{\boldsymbol{m}}^{(a)}$$

- Useful relation

$$\overline{\Psi}^+_{i,\boldsymbol{m},a}(u) = \Psi^+_{l-i+1,\boldsymbol{m},l-a+2}(-(-1)^l u), \qquad \bar\lambda_{\boldsymbol{m},a} = \iota(\lambda_{\boldsymbol{m},l-a+2})$$

  where

$$\iota\colon \mathfrak{h}^* \to \mathfrak{h}^*, \qquad \omega_i \in \mathfrak{h}^*, \qquad \langle \omega_i, H_j\rangle = \delta_{ij}, \qquad \iota(\omega_i) = \omega_{l-i+1}$$

Highest ℓ-weights

- For representations $(\theta_a)_\zeta$

$$\lambda_{\mathbf{0},1} = -(l+1)\,\omega_1$$

$$\Psi^+_{i,\mathbf{0},1}(u) = \begin{cases} q^{-l-1}\left(1 - q^{-l}\,\zeta^s\,u\right)^{-1}, & i = 1 \\ 1, & i = 2,\ldots,l \end{cases}$$

$$\lambda_{\mathbf{0},a} = (l-a+1)\,\omega_{a-1} - (l-a+2)\,\omega_a$$

$$\Psi^+_{i,\mathbf{0},a}(u) = \begin{cases} 1, & i = 1,\ldots,a-2 \\ q^{l-a+1}\left(1 - q^{-l+a}\,\zeta^s\,u\right), & i = a-1 \\ q^{-l+a-2}\left(1 - q^{-l+a-1}\,\zeta^s\,u\right)^{-1}, & i = a \\ 1, & i = a+1,\ldots,l \end{cases}$$

$$\lambda_{\mathbf{0},l+1} = 0$$

$$\Psi^+_{i,\mathbf{0},l+1}(u) = \begin{cases} 1, & i = 1,\ldots,l-1 \\ 1 - q\,\zeta^s\,u, & i = l \end{cases}$$

Highest ℓ-weights

- For representations $(\bar{\theta}_a)_\zeta$

$$\bar{\lambda}_{\mathbf{0},1} = 0$$

$$\bar{\Psi}^+_{i,\mathbf{0},1}(u) = \begin{cases} 1 + (-1)^l q\, \zeta^s u, & i = 1 \\ 1, & i = 2, \ldots, l \end{cases}$$

$$\bar{\lambda}_{\mathbf{0},a} = -a\,\omega_{a-1} + (a-1)\,\omega_a$$

$$\bar{\Psi}^+_{i,\mathbf{0},a}(u) = \begin{cases} 1, & i = 1, \ldots, a-2 \\ q^{-a}\left(1 + (-1)^l q^{-a+1} \zeta^s u\right)^{-1}, & i = a-1 \\ q^{a-1}\left(1 + (-1)^l q^{-a+2} \zeta^s u\right), & i = a \\ 1, & i = a+1, \ldots, l \end{cases}$$

$$\bar{\lambda}_{\mathbf{0},l+1} = -(l+1)\,\omega_l$$

$$\bar{\Psi}^+_{i,\mathbf{0},l+1}(u) = \begin{cases} 1, & i = 1, \ldots, l-1 \\ q^{-l-1}\left(1 + (-1)^l q^{-l} \zeta^s u\right)^{-1}, & i = l \end{cases}$$

Oscillator versus prefundamental representations I.

- Representations $(\theta_a)_\zeta$

$$(\theta_1)_\zeta \cong L_{\xi_1} \otimes L^-_{1,q^{-l}\zeta^s}$$

$$(\theta_a)_\zeta \cong L_{\xi_a} \otimes (L^+_{a-1,q^{-l+a}\zeta^s} \mathbin{\overline{\otimes}} L^-_{a,q^{-l+a-1}\zeta^s}), \qquad a = 2, \dots, l$$

$$(\theta_{l+1})_\zeta \cong L^+_{l,q\,\zeta^s}$$

$$\xi_a = (l-a+1)\,\omega_{a-1} - (l-a+2)\,\omega_a$$

- Representations $(\bar{\theta}_a)_\zeta$

$$(\bar{\theta}_1)_\zeta \cong L^+_{1,(-1)^{l+1}q\zeta^s}$$

$$(\bar{\theta}_a)_\zeta \cong L_{\bar{\xi}_a} \otimes (L^-_{a-1,(-1)^{l-1}\,q^{-a+1}\zeta^s} \mathbin{\overline{\otimes}} L^+_{a,(-1)^{l+1}\,q^{-a+1}\zeta^s}), \qquad a = 2, \dots, l$$

$$(\bar{\theta}_{l+1})_\zeta \cong L_{\bar{\xi}_{l+1}} \otimes L^-_{l,(-1)^{l+1}\,q^{-l}\,\zeta^s}$$

$$\bar{\xi}_a = -a\,\omega_{a-1} + (a-1)\,\omega_a$$

Oscillator versus prefundamental representations II.

- $L^{\pm}_{i,z}$ through $(\theta_a)_\zeta$

$$L_{\xi_i^+} \otimes L^+_{i,\zeta^s} \cong (\theta_{i+1})_{q^{l-i-1}\zeta^s} \,\overline{\otimes}\, (\theta_{i+2})_{q^{l-i-3}\zeta^s} \,\overline{\otimes} \dots \overline{\otimes}\, (\theta_{l+1})_{q^{-l+i-1}\zeta^s}$$

$$L_{\xi_i^-} \otimes L^-_{i,\zeta^s} \cong (\theta_1)_{q^{l+i-1}\zeta^s} \,\overline{\otimes}\, (\theta_2)_{q^{l+i-3}\zeta^s} \,\overline{\otimes} \dots \overline{\otimes}\, (\theta_i)_{q^{l-i+1}\zeta^s}$$

$$\xi_i^+ = (l-i)\,\omega_i - 2\sum_{j=i+1}^{l} \omega_j, \qquad \xi_i^- = -2\sum_{j=1}^{i-1}\omega_j - (l-i+2)\,\omega_i$$

- $L^{\pm}_{i,z}$ through $(\bar{\theta}_a)_\zeta$

$$L_{\xi_i^+} \otimes L^+_{i,\zeta^s} \cong (\bar{\theta}_1)_{(-1)^{l-1}\,q^{-i}\zeta^s} \,\overline{\otimes}\, (\bar{\theta}_2)_{(-1)^{l-1}\,q^{2-i}\zeta^s} \,\overline{\otimes} \dots \overline{\otimes}\, (\bar{\theta}_i)_{(-1)^{l-1}\,q^{i-2}\zeta^s}$$

$$L_{\xi_i^-} \otimes L^-_{i,\zeta^s} \cong (\bar{\theta}_{i+1})_{(-1)^{l-1}\,q^{i}\zeta^s} \,\overline{\otimes}\, (\bar{\theta}_{i+2})_{(-1)^{l-1}\,q^{i+2}\zeta^s} \,\overline{\otimes} \dots \overline{\otimes}\, (\bar{\theta}_{l+1})_{(-1)^{l-1}\,q^{2l-i}\zeta^s}$$

$$\xi_i^+ = -2\sum_{j=1}^{i-1}\omega_j + (i-1)\,\omega_i, \qquad \xi_i^- = -(i+1)\,\omega_i - 2\sum_{j=i+1}^{l}\omega_j$$

$(\widetilde{V}^{\lambda})_{\zeta}[\xi]$ versus $\bigotimes_{a=1}^{l+1}(W_a)_{\zeta_a}$

- We first note for $(W_1)_{\zeta_1} \otimes \cdots \otimes (W_{l+1})_{\zeta_{l+1}}$

$$\Psi_i^+(u) = q^{-2}\,\frac{1-q^{-l+i+1}\,\zeta_{i+1}^s\,u}{1-q^{-l+i-1}\,\zeta_i^s\,u}, \qquad i=1,\ldots,l$$

- Further, restricting $(\widetilde{V}^{\lambda})_{\zeta}$ to $\mathrm{U}_q(\mathfrak{b}^+)$

$$\Psi_i^+(u) = q^{\lambda_i-\lambda_{i+1}}\,\frac{1-q^{2\lambda_{i+1}-i+1}\,\zeta^s\,u}{1-q^{2\lambda_i-i+1}\,\zeta^s\,u}, \qquad i=1,\ldots,l$$

- In particular

$$\Psi_{1,\mathbf{m}}^+(u) = q^{\lambda_1-\lambda_2-2m_{12}-\sum_{i=3}^{l+1}(m_{1i}-m_{2i})} \times \frac{\left(1-q^{2\lambda_1-2\sum_{i=3}^{l+1}m_{1i}+2}\,u\right)\left(1-q^{2\lambda_2-2\sum_{i=3}^{l+1}m_{2i}}\,u\right)}{\left(1-q^{2\lambda_1-2\sum_{i=2}^{l+1}m_{1i}}\,u\right)\left(1-q^{2\lambda_1-2\sum_{i=2}^{l+1}m_{1i}+2}\,u\right)}$$

- The isomorphism at

$$\zeta_i = q^{2\langle\lambda+\rho,K_i\rangle/s}\,\zeta, \qquad \langle\rho,K_i\rangle = \frac{l}{2}-i+1, \qquad \xi = -\sum_{i=1}^{l}(\lambda_i-\lambda_{i+1}+2)\omega_i$$

- **The q-oscillator representations are no less fundamental than the prefundamental ones**

- $\mathfrak{sl}_{l+1}$: $2l$ **prefundamental reps vs** $2l+2$ q**-oscillator reps**

- $\mathfrak{sl}_2$: **a special case** $-$ **only** 2 **reps of both kinds**

- **Relations between highest ℓ-weights replicate functional relations between universal integrability objects**

Thank you!

Appendix

$\mathrm{U}_q(\mathfrak{b}^+)$-module relations

- Restricting $\widetilde{V}^\lambda \to \left.\widetilde{V}^\lambda\right|_{\mathrm{U}_q(\mathfrak{b}^+)}$

$$q^{\nu h_0}\, v_{\boldsymbol{m}} = q^{\nu[\lambda_{l+1}-\lambda_1+\sum_{i=2}^{l}(m_{1i}+m_{i,l+1})+2m_{1,l+1}]}\, v_{\boldsymbol{m}}$$

$$q^{\nu h_i}\, v_{\boldsymbol{m}} = q^{\nu[\lambda_i-\lambda_{i+1}+\sum_{k=1}^{i-1}(m_{ki}-m_{k,i+1})-2m_{i,i+1}-\sum_{k=i+2}^{l+1}(m_{ik}-m_{i+1,k})]} v_{\boldsymbol{m}}$$

$$e_0\, v_{\boldsymbol{m}} = \zeta^{s_0}\, q^{\lambda_1+\lambda_{l+1}+\sum_{i=2}^{l} m_{i,l+1}}\, v_{\boldsymbol{m}+\epsilon_{1,l+1}}$$

$$\begin{aligned} e_i\, v_{\boldsymbol{m}} = {} & \zeta^{s_i}[\lambda_i - \lambda_{i+1} - \sum_{j=i+2}^{l+1}(m_{ij} - m_{i+1,j}) - m_{i,i+1} + 1]_q\, [m_{i,i+1}]_q\, v_{\boldsymbol{m}-\epsilon_{i,i+1}} \\ & + \zeta^{s_i} q^{\lambda_i-\lambda_{i+1}-2m_{i,i+1}-\sum_{j=i+2}^{l+1}(m_{ij}-m_{i+1,j})} \\ & \qquad \times \sum_{j=1}^{i-1} q^{\sum_{k=j+1}^{i-1}(m_{ki}-m_{k,i+1})}\, [m_{j,i+1}]_q\, v_{\boldsymbol{m}-\epsilon_{j,i+1}+\epsilon_{ji}} \\ & - \zeta^{s_i} \sum_{j=i+2}^{l+1} q^{-\lambda_i+\lambda_{i+1}-2+\sum_{k=j}^{l+1}(m_{ik}-m_{i+1,k})}\, [m_{ij}]_q\, v_{\boldsymbol{m}-\epsilon_{ij}+\epsilon_{i+1,j}} \end{aligned}$$

Degenerations of the shifted $\mathrm{U}_q(\mathfrak{b}^+)$-modules

- Let $\widetilde{\varphi}^{\lambda}_{\zeta}$ be a representation of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$, and $\xi \in \widetilde{\mathfrak{h}}^*_{l+1}$. Then

$$\widetilde{\varphi}^{\lambda}_{\zeta}[\xi](e_i) = \widetilde{\varphi}^{\lambda}_{\zeta}(e_i), \qquad \widetilde{\varphi}^{\lambda}_{\zeta}[\xi](q^x) = q^{\langle \xi, x\rangle}\, \widetilde{\varphi}^{\lambda}_{\zeta}(q^x)$$

  is a *shifted representation* representation of $\mathrm{U}_q(\mathfrak{b}^+)$

- Universal integrability objects after the module shift

$$\mathcal{T}_{\widetilde{\varphi}^{\lambda}[\xi]}(\zeta) = \mathcal{T}_{\widetilde{\varphi}^{\lambda}}(\zeta)\, q^{\sum_{i=0}^{l} \langle \xi, h_i\rangle (h_i + \phi_i)/(l+1)}$$

- Specific shifts $\xi \in \widetilde{\mathfrak{h}}^*_{l+1}$

$$\langle \xi, h_0 \rangle = \lambda_1 - \lambda_{l+1}, \qquad \langle \xi, h_i \rangle = -\lambda_i + \lambda_{i+1}, \qquad i = 1, \ldots, l$$

- Redefine the basis vectors $v_{\boldsymbol{m}} \to c_{\boldsymbol{m}} v_{\boldsymbol{m}}$ and take the limit

$$\lambda_i - \lambda_{i+1} \to -\infty, \qquad i = 1, \ldots, l, \qquad\qquad \widetilde{V}^{\lambda}\Big|_{\mathrm{U}_q(\mathfrak{b}^+)} \to \widetilde{V}^{\infty}$$

- There is maximal submodule $\widetilde{V}_{\max} \subset \widetilde{V}^{\infty}$: obtain irreducible module W'

$$\widetilde{V}_{\max} \subset \widetilde{V}^{\infty} : \qquad W' \cong \widetilde{V}^{\infty} / \widetilde{V}_{\max}$$

$\mathrm{U}_q(\mathfrak{b}^+)$-submodules and q-oscillators

- ρ' and W' for $\mathrm{U}_q(\mathfrak{b}^+)$

$$q^{\nu h_0}\, v_{\boldsymbol{m}} = q^{\nu(2m_1 + \sum_{j=2}^{l} m_j)}\, v_{\boldsymbol{m}}, \qquad q^{\nu h_l}\, v_{\boldsymbol{m}} = q^{-\nu(2m_l + \sum_{j=1}^{l-1} m_j)}\, v_{\boldsymbol{m}}$$

$$q^{\nu h_i}\, v_{\boldsymbol{m}} = q^{\nu(m_{i+1} - m_i)} v_{\boldsymbol{m}}, \qquad i = 1, \dots, l-1$$

$$e_0\, v_{\boldsymbol{m}} = q^{\sum_{j=2}^{l} m_j}\, v_{\boldsymbol{m}+\epsilon_1}, \qquad e_l\, v_{\boldsymbol{m}} = -\kappa_q^{-1}\, q^{m_l}\, [m_l]_q\, v_{\boldsymbol{m}-\epsilon_l}$$

$$e_i\, v_{\boldsymbol{m}} = -q^{m_i - m_{i+1} - 1}\, [m_i]_q\, v_{\boldsymbol{m}-\epsilon_i+\epsilon_{i+1}}, \qquad i = 1, \dots, l-1$$

- Osc_q is a unital associative $\mathbb{C}$-algebra with generators $b^\dagger, b, q^{\nu N}, \nu \in \mathbb{C}$

$$q^0 = 1, \qquad q^{\nu_1 N} q^{\nu_2 N} = q^{(\nu_1+\nu_2)N}$$

$$q^{\nu N} b^\dagger q^{-\nu N} = q^\nu b^\dagger, \qquad q^{\nu N} b q^{-\nu N} = q^{-\nu} b$$

$$b^\dagger b = [N]_q, \qquad b b^\dagger = [N+1]_q$$

$\{(b^\dagger)^{k+1} q^{\nu N},\, b^{k+1} q^{\nu N},\, q^{\nu N} \mid k \in \mathbb{Z}_+, \nu \in \mathbb{C}\}$ form a basis of Osc_q

Representations of Osc_q

- W^+ , χ^+ : The relations

$$q^{\nu N} v_m = q^{\nu m} v_m$$

$$b^\dagger v_m = v_{m+1}, \qquad b\, v_m = [m]_q v_{m-1}$$

where $v_{-1} = 0$, endow the free vector space generated by $\{v_0, v_1, \ldots\}$ with the structure of an Osc_q-module

- W^- , χ^- : The relations

$$q^{\nu N} v_m = q^{-\nu(m+1)} v_m$$

$$b\, v_m = v_{m+1}, \qquad b^\dagger\, v_m = -[m]_q v_{m-1}$$

where $v_{-1} = 0$, endow the free vector space generated by $\{v_0, v_1, \ldots\}$ with the structure of an Osc_q-module

- We consider the algebra $\mathrm{Osc}_q \otimes \ldots \otimes \mathrm{Osc}_q = \mathrm{Osc}_q^{\otimes\, l}$ and define

$$b_i = 1 \otimes \ldots \otimes b \otimes \ldots \otimes 1, \qquad b_i^\dagger = 1 \otimes \ldots \otimes b^\dagger \otimes \ldots \otimes 1$$

$$q^{\nu N_i} = 1 \otimes \ldots \otimes q^{\nu N} \otimes \ldots \otimes 1$$

Interpretation in terms of q-oscillators

- Consider W' and ρ' in terms of the q-oscillators

$$
\begin{aligned}
&q^{\nu h_0}\, v_{\boldsymbol{m}} = q^{\nu(2N_1+\sum_{i=2}^{l} N_i)}\, v_{\boldsymbol{m}}, && q^{\nu h_l}\, v_{\boldsymbol{m}} = q^{-\nu(2N_l+\sum_{i=1}^{l-1} N_i)}\, v_{\boldsymbol{m}} \\
&q^{\nu h_i}\, v_{\boldsymbol{m}} = q^{\nu(N_{i+1}-N_i)}\, v_{\boldsymbol{m}}, && i = 1, \ldots, l-1 \\
&e_0\, v_{\boldsymbol{m}} = b_1^{\dagger}\, q^{\sum_{i=2}^{l} N_i}\, v_{\boldsymbol{m}}, && e_l\, v_{\boldsymbol{m}} = -\kappa_q^{-1}\, b_l\, q^{N_l}\, v_{\boldsymbol{m}} \\
&e_i\, v_{\boldsymbol{m}} = -b_i\, b_{i+1}^{\dagger}\, q^{N_i-N_{i+1}-1}\, v_{\boldsymbol{m}}, && i = 1, \ldots, l-1
\end{aligned}
$$

- Define a homomorphism $\rho : \mathrm{U}_q(\mathfrak{b}^+) \to \mathrm{Osc}_q^{\otimes l}$ by

$$
\begin{aligned}
&\rho(q^{\nu h_0}) = q^{\nu(2N_1+\sum_{i=2}^{l} N_i)}, && \rho(q^{\nu h_l}) = q^{-\nu(2N_l+\sum_{i=1}^{l-1} N_i)} \\
&\rho(q^{\nu h_i}) = q^{\nu(N_{i+1}-N_i)}, && i = 1, \ldots, l-1 \\
&\rho(e_0) = b_1^{\dagger}\, q^{\sum_{i=2}^{l} N_i}, && \rho(e_l) = -\kappa_q^{-1}\, b_l\, q^{N_l} \\
&\rho(e_i) = -b_i\, b_{i+1}^{\dagger}\, q^{N_i-N_{i+1}-1}, && i = 1, \ldots, l-1
\end{aligned}
$$

Cartan–Weyl generators of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

- System of positive roots of $\widehat{\mathfrak{sl}}_{l+1}$

$$\widehat{\triangle}_+ = \{\gamma + k\delta \mid \gamma \in \triangle_+,\, k \in \mathbb{Z}_+\} \\ \cup \{m\delta \mid m \in \mathbb{N}\} \cup \{(\delta - \gamma) + n\delta \mid \gamma \in \triangle_+,\, n \in \mathbb{Z}_+\}$$

- Normal order

$$\gamma + k\delta \prec m\delta \prec (\delta - \gamma) + n\delta, \qquad \gamma \in \triangle_+, \qquad k, m, n \in \mathbb{Z}_+$$

- Higher root vectors

$$e_{\gamma+n\delta} = [2]_q^{-1}[e_{\gamma+(n-1)\delta}, e'_{\delta,\gamma}]_q$$

$$e_{(\delta-\gamma)+n\delta} = [2]_q^{-1}[e'_{\delta,\gamma}, e_{(\delta-\gamma)+(n-1)\delta}]_q$$

$$e'_{n\delta,\gamma} = [e_{\gamma+(n-1)\delta}, e_{\delta-\gamma}]_q$$

$$-\kappa_q\, e_{\delta,\gamma}(u) = \log(1 - \kappa_q\, e'_{\delta,\gamma}(u)), \qquad e'_{\delta,\gamma}(u) = \sum_{n\in\mathbb{N}} e'_{n\delta,\gamma}\, u^n$$

Specific monodromy and L-operators

- Monodromy operator $\quad M(\zeta|\eta) = (\varepsilon_\zeta \otimes (\varphi^{(1,0,\dots,0)})_\eta)(\mathcal{R})$

$$
\begin{aligned}
&\mathbb{M}(\zeta)_{ij} = -\zeta^{s - s_{ij}}\, \kappa_q\, q^{K_i}\, F_{ij}, && 1 \le i < j \le l+1 \\
&\mathbb{M}(\zeta)_{ii} = q^{-K_i} - \zeta^{s}\, q^{K_i}, && i = 1, \dots, l+1 \\
&\mathbb{M}(\zeta)_{ij} = -\zeta^{s_{ji}}\, \kappa_q\, E_{ji}\, q^{-K_j}, && 1 \le j < i \le l+1
\end{aligned}
$$

- L-operator $\quad L(\zeta|\eta) = (\rho_\zeta \otimes (\varphi^{(1,0,\dots,0)})_\eta)(\mathcal{R})$

$$
\begin{aligned}
&\mathbb{L}(\zeta)_{i,l+1} = -\zeta^{s - s_{i,l+1}}\, \kappa_q\, b_i^\dagger\, q^{N_{1i} - N_i + i - 1}, && i = 1, \dots, l \\
&\mathbb{L}(\zeta)_{ij} = 0, && i < j < l+1 \\
&\mathbb{L}(\zeta)_{ij} = \zeta^{s_{ji}}\, \kappa_q\, b_j\, b_i^\dagger\, q^{N_j + N_{ji} - N_i + i - j - 2}, && 1 < i - j < l \\
&\mathbb{L}(\zeta)_{i+1,i} = \zeta^{s_i}\, \kappa_q\, b_i\, b_{i+1}^\dagger\, q^{2N_i - N_{i+1} - 1}, && i = 1, \dots, l-1 \\
&\mathbb{L}(\zeta)_{l+1,i} = \zeta^{s_{i,l+1}}\, b_i\, q^{N_i + N_{i,l+1} + l - i} && i = 1, \dots, l
\end{aligned}
$$

$$
\mathbb{L}(\zeta)_{ii} = q^{N_i}, \quad i = 1, \dots, l, \qquad \mathbb{L}(\zeta)_{l+1,l+1} = q^{-N_{1,l+1}} - \zeta^{s}\, q^{N_{1,l+1} + l + 1}
$$