New deformations of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric mechanics

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Supersymmetric Quantum Mechanics (SQM) (Witten, 1981) is the simplest ($d = 1$) supersymmetric theory:

- Catches the basic features of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc;
- Originally, $\mathcal{N} = 2$: $\{Q, \bar{Q}\} = 2H\,,\ Q^2 = \bar{Q}^2 = 0,\ [Q, H] = [\bar{Q}, H] = 0\,.$
- Extended $\mathcal{N} > 2, d = 1$ SUSY is specific: dualities between various supermultiplets (Gates Jr. & Rana, 1995, Pashnev & Toppan, 2001), nonlinear "cousins" of off-shell linear multiplets (I., Krivonos, Lechtenfeld, 2003, 2004), etc.
- $\mathcal{N} = 4$ SQM: $\{Q_\alpha, \bar{Q}^\beta\} = 2\delta_\alpha^\beta H\,, \alpha = 1, 2,$ is of special interest. In particular, a subclass of $\mathcal{N} = 4$ SQM models have as their bosonic target, Hyper-Kähler (HK) manifolds.

In this Talk, two different types of deformations of $\mathcal{N} = 4$ SQM models will be presented.

From deformed $\mathcal{N}=4$ SQM to its $\mathcal{N}=8$ extensions

The first type of deformed SQM arises while choosing some semi-simple supergroups instead of higher-rank $d=1$ super-Poincare:

A. Standard extension:

$$(\mathcal{N}=2\,,\ d=1) \quad \Rightarrow \quad (\mathcal{N}>2\,,\ d=1\ \text{Poincaré})\,,$$

B. Non-standard extension:

$$(\mathcal{N}=2\,,\ d=1) \equiv u(1|1) \quad \Rightarrow \quad su(2|1)\ \subset\ su(2|2)\ \subset\ \ldots\,.$$

In the chain B, the closure of supercharges contains, besides H, also internal symmetry generators. They commute with H, but not with the supercharges. The deformed $\mathcal{N}=4$ SQM is associated with $su(2|1)$:

$$\{Q^i,\bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$
$$\left[I^i_j,\bar{Q}_l\right] = \frac{1}{2}\delta^i_j\bar{Q}_l - \delta^i_l\bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[F,\bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k.$$

The parameter m is a deformation parameter: when $m\to 0$, the standard $\mathcal{N}=4, d=1$ super-Poincare is restored.

- The simplest models with world-line realization of $su(2|1)$ were considered in (Belucci & Nersessian, 2003, 2004; Römelsberger, 2006, 2007) and in (Smilga, 2004) (named there “week $d = 1$ suprsymmetry”). The corresponding world-line multiplets were $(2, 4, 2)$ and $(1, 4, 3)$.

- The systematic superfield approach to $su(2|1)$ supersymmetry was worked out in (I. & Sidorov, 2014, 2016; I., Sidorov & Toppan, 2015). The models built on the multiplets $(1, 4, 3)$, $(2, 4, 2)$ and $(4, 4, 0)$ were studied at the classical and quantum levels. .

- Recently, $su(2|1)$ invariant versions of super Calogero-Moser systems were constructed and quantized (Fedoruk & I., 2017; Fedoruk, I., Lechtenfeld & Sidorov, 2017).

The common features of all these models are:

- The oscillator-type Lagrangians for the bosonic fields, with m^2 as the oscillator strength.
- The appearance of the Wess-Zumino type terms for the bosonic fields, of the type $\sim im(\dot{z}\bar{z} - z\dot{\bar{z}})$.
- At the lowest energy levels, wave functions form atypical $su(2|1)$ multiplets, with unequal numbers of the bosonic and fermionic states.

Deformed $\mathcal{N}=8$ mechanics

The flat $\mathcal{N}=8$ superalgebra,

$$\{Q_A, Q_B\} = 2\delta_{AB}H\,, \quad A, B = 1, \ldots, 8\,,$$

admits two deformations with the minimal number of extra bosonic generators.

A. Superalgebra $su(2|2)$ (I., Lechtenfeld, Sidorov, 2016):

$$\left\{Q^{ia}, S^{jb}\right\} = 2im\left(\varepsilon^{ab}L^{ij} - \varepsilon^{ij}R^{ab}\right) + 2\,\varepsilon^{ab}\varepsilon^{ij}C,$$
$$\left\{Q^{ia}, Q^{jb}\right\} = 2\,\varepsilon^{ij}\varepsilon^{ab}(H + C_1), \qquad \left\{S^{ia}, S^{jb}\right\} = 2\,\varepsilon^{ij}\varepsilon^{ab}(H - C_1).$$

In the limit $m \to 0$ a centrally extended flat $\mathcal{N}=8$ superalgebra is reproduced, with two extra central charges C and C_1 and $SO(8)$ automorphisms broken to $SU(2)\times SU(2)$.

B. Superalgebra $su(4|1)$ (I., Lechtenfeld, Sidorov, in preparation):

$$\left\{Q^I, \bar{Q}_J\right\} = 2m\,L^I_J + 2\delta^I_J\mathcal{H}, \quad I, J = 1, \ldots, 4\,,$$
$$\left[\mathcal{H}, Q^K\right] = -\,\frac{3m}{4}\,Q^K, \qquad \left[\mathcal{H}, \bar{Q}_L\right] = \frac{3m}{4}\,\bar{Q}_L\,.$$

$SU(4)$ automorphisms (instead of $SO(8)$ or $SU(2)\times SU(2)$).

- In the case A we constructed, by analogy with $SU(2|1)$, the world-line superfield techniques and presented a few $SU(2|2)$ SQM models as deformations of flat $\mathcal{N}=8$ models. These are based on the off-shell $SU(2|2)$ multiplets (**3, 8, 5**), (**4, 8, 4**) and (**5, 8, 3**).

- The actions for the $SU(2|2)$ multiplet (**5, 8, 3**) are massive deformations of those for the same multiplet in the flat case (Ivanov & Smilga, 2004). A particular class of such actions yields in the bosonic sector,

$$\mathcal{L}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = g\big[\dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} - m^2 v_{ij}v^{ij} + \frac{1}{2}B_{ab}B^{ab}\big] - \frac{i}{2}mg\big(\dot{\bar{z}}z - \dot{z}\bar{z}\big),$$
$$g(z,\bar{z}) = f''(z) + \bar{f}''(\bar{z}). \qquad (1)$$

It reveals the special Kähler geometry in the $(z,\bar{z})$ sector. Another class of actions enjoys superconformal $OSp(4^*|4)$ invariance

$$\mathcal{L}^{\text{conf.}}_{(\mathbf{5},\mathbf{8},\mathbf{3})}\,|_{\text{bos}} = \left(v_{ij}v^{ij} + z\bar{z}\right)^{-3/2}\left[\dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} + \frac{1}{2}B^{ab}B_{ab} - m^2\left(v_{ij}v^{ij} + z\bar{z}\right)\right].$$

- Constructing the superconformal actions is based on the fact that the superconformal group $OSp(4^*|4)$ is a closure of its two $SU(2|2)$ subgroups: with the parameters m and $-m$. So any $SU(2|2)$ invariant action involving only even powers of m is automatically superconformal. Based on this, the general $SU(2|2)$ action of (**3, 8, 5**) was shown to be superconformal.

- Not all of the admissible multiplets of the flat $\mathcal{N}=8$ SQM have $SU(2|2)$ analogs. It is most important that the so called "root" $\mathcal{N}=8$ multiplet (**8, 8, 0**) does not have.
- Meanwhile, all other flat $\mathcal{N}=8$ multiplets and their invariant actions can be obtained from the root one and its general action through the appropriate covariant truncations (or Hamiltonian reductions, in the Hamiltonian formalism) (I., Lechtenfeld, Sutulin, 2008). How to construct a deformed version of the (**8, 8, 0**) multiplet?
- It becomes possible in the models based on world-line realizations of the supergroup $SU(4|1)$. This multiplet is described by a chiral $SU(4|1)$ superfield

$$\Phi\left(t_{\rm L},\theta_I\right)=\phi+\sqrt{2}\,\theta_K\chi^K+\theta_I\theta_J A^{IJ}+\frac{\sqrt{2}}{3}\,\theta_I\theta_J\theta_K\xi^{IJK}+\frac{1}{4}\,\varepsilon^{IJKL}\,\theta_I\theta_J\theta_K\theta_L B\,,$$

with the additional $SU(4|1)$ covariant constraints on the component fields

$$A^{IJ}=\sqrt{2}(i\dot{y}^{IJ}-\frac{m}{2}\,y^{IJ}),\ \xi^{IJK}=-\,\varepsilon^{IJKL}(i\dot{\bar{\chi}}_L-\frac{5m}{4}\,\bar{\chi}_L),\ B=\frac{2}{3}(\ddot{\bar{\phi}}+2im\dot{\bar{\phi}})$$

- We end up with just $8=2+6$ real bosonic fields (ϕ, y^{IJ}) in the $SU(4)$ representation $(\underline{1}\oplus\underline{6})$ and 4 complex fermionic fields χ^L in the fundamental of $SU(4)$.

- The invariant action has the very simple form

$$S_{(\mathbf{8},\mathbf{8},\mathbf{0})} = \int dt\, \mathcal{L}_{\rm SK} = \int d\zeta_{\rm L}\, K(\Phi) + \int d\zeta_{\rm R}\, \bar{K}(\bar{\Phi})$$

where, in the bosonic limit,

$$\mathcal{L}^{\rm bos}_{(\mathbf{8},\mathbf{8},\mathbf{0})} = g\big(\dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} - \frac{m^2}{8}\,y^{IJ}y_{IJ}\big)$$
$$- \frac{im}{4}\big(\dot{\phi}\,\partial_{\phi}g - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g\big)y^{IJ}y_{IJ} + 2im\big(\dot{\phi}\,\partial_{\bar{\phi}}\bar{K} - \dot{\bar{\phi}}\,\partial_{\phi}K\big),$$

and

$$g \sim \partial_{\phi}\partial_{\phi}K(\phi) + \partial_{\bar{\phi}}\partial_{\bar{\phi}}\bar{K}(\bar{\phi}).$$

- Besides this action, there were constructed two more invariant actions which are not equivalent to each other. One of them depends only on m^2 and exhibits the relevant superconformal symmetry $OSp(8|2)$ which also can be represented as a closure of its two $SU(4|1)$ subgroups intersecting over $SU(4)$.
- Also, the actions for the $SU(4|1)$ multiplets (**6, 8, 2**) and (**7, 8, 1**) were given. It is likely that the rest of other multiplets missing in the $SU(2|2)$ case can also be constructed within the $SU(4|1)$ superfield formalism.
- It was also found that there exists another (twisted) multiplet (**8, 8, 0**), with the bosonic fields in $\underline{4}$ of $SU(4)$.

Prospects

- Possible applications -in supersymmetric matrix models (Berenstein, Maldacena, Nastace, 2002; Dasgupta, Sheikh-Jabbari, Van Raamsdonk, 2002, and others). They possess $SU(4|2)$ invariance, and so $SU(2|2) \subset SU(4|2)$ and $SU(4|1) \subset SU(4|2)$ SQM can describe some important truncations of these models.

- Our superfield approach gives non-trivial $SU(2|2)$ and $SU(4|1)$ invariant interactions and so could provide quantum corrections to the supermatrix theory actions, defining a kind of effective actions for these systems.

- The basic on-shell $SU(4|2)$ multiplet (**9, 16**) of matrix models could be constructed from few $SU(2|2)$ or $SU(4|1)$ multiplets.

So much for this sort of deformations!

QK $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Another type of deformations of $\mathcal{N} = 4$ SQM models starts from the general Hyper-Kähler (HK) subclass of the latter. The deforrmed models are $\mathcal{N} = 4$ supersymetrization of the Quaternion-Kähler (QK) $d = 1$ sigma models (I. & Mezincescu, 2017).

Both HK and QK $\mathcal{N} = 4$ SQM models can be derived from $\mathcal{N} = 4, d = 1$ harmonic superspace approach (I. & Lechtenfeld, 2003), which:

- Allows one to understand interrelations between various $\mathcal{N} = 4$ *SQM* models via the manifestly $\mathcal{N} = 4$ covariant gauging procedure (Delduc, & I., 2006, 2007);
- Provides new $\mathcal{N} = 4$ superextensions of Calogero-type models (Fedoruk, I., Lechtenfeld, 2008 - 2010);
- Ensures an unambiguous construction of $\mathcal{N} = 4$ SQM models with the Lorentz-force type couplings to an external non-abelian gauge field (I., Konyushikhin, Smilga, 2009, 2010);
- Only within the $d = 1$ HSS framework it is possible to construct off-shell $\mathcal{N} = 4$ SQM models associated with general HK bosonic manifolds (Delduc, I., 2010).

HK manifolds are bosonic targets of sigma models with **rigid** $\mathcal{N}=2, d=4$ SUSY (Alvarez-Gaumé, Freedman, 1980). After coupling these models to **local** $\mathcal{N}=2, d=4$ SUSY in the supergravity framework the target spaces are deformed into the so called Quaternion-Kähler (QK) manifolds (Bagger, Witten, 1983). QK manifolds are also $4n$ dimensional, but their holonomy group is a subgroup of $Sp(1)\times Sp(n)$. The deformation parameter is just Einstein constant κ, and in the "flat" limit $\kappa\to 0$, the appropriate HK manifolds are recovered.

What about $\mathcal{N}=4$ Quaternion-Kähler SQM? Nobody succeeded in constructing such models before, seemingly due to difficulties of accounting for the supergravity quantities in the quantum-mechanical context. The main problem was how to ensure, in one or another way, a local supersymmetry and local $SU(2)$ automorphism symmetry.

In this part of the talk, which is based on a recent paper with Luca Mezincescu (University of Miami), it will be shown how to construct $\mathcal{N}=4$ SQM with an arbitrary QK bosonic target. Like in constructing $\mathcal{N}=4$ HK SQM, the basic tool is $d=1$ harmonic superspace.

Harmonic $\mathcal{N}=4, d=1$ superspace

- Ordinary $\mathcal{N}=4, d=1$ superspace:

$$(t, \theta^i, \bar\theta_k), \quad i, k, = 1, 2;$$

- Harmonic extension:

$$(t, \theta^i, \bar\theta_k) \quad \Rightarrow \quad (t, \theta^i, \bar\theta_k, u_j^\pm),\ u^{+i}u_i^- = 1,\ u_i^\pm \in SU(2)_{Aut}.$$

- Analytic basis:

$$(t_A, \theta^+, \bar\theta^+, u_k^\pm, \theta^-, \bar\theta^-) \equiv (\zeta, u^\pm, \theta^-, \bar\theta^-)$$

$$\theta^\pm = \theta^i u_i^\pm,\ \bar\theta^\pm = \bar\theta^k u_k^\pm,\ \ t_A = t + i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)$$

- Analytic superspace and superfields:

$$D^+ = \frac{\partial}{\partial\theta^-},\ \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}, \quad D^+\Phi = \bar D^+\Phi = 0 \ \Rightarrow\ \Phi = \Phi(\zeta, u^\pm)$$

- Harmonic derivatives:

$$D^{\pm\pm} = u_\alpha^\pm \frac{\partial}{\partial u_\alpha^\mp} + \theta^\pm \frac{\partial}{\partial\theta^\mp} + \bar\theta^\pm \frac{\partial}{\partial\bar\theta^\mp} + 2i\theta^\pm\bar\theta^\pm \frac{\partial}{\partial t_{\mathrm{A}}},$$

$[D^+, D^{++}] = [\bar D^+, D^{++}] = 0 \quad \Rightarrow \quad D^{++}\Phi(\zeta, u^\pm)$ is analytic

Basic $\mathcal{N}=4, d=1$ multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$

- Described off-shell by an analytic superfield $q^{+a}(\zeta,u)$:

$$(4,4,0) \quad \Longleftrightarrow \quad q^{+a}(\zeta,u) \propto (f^{ia},\chi^a,\bar\chi^a),\; a=1,2,$$

$$(a)\; D^+q^{+a}=0\; (Grassmann\;\; analyticity),$$
$$(b)\; D^{++}q^{+a}=0\; (Harmonic\;\; analyticity),$$

$$(a)+(b) \quad \Longrightarrow \quad q^{+a}=f^{ka}u^+_k+\theta^+\chi^a-\bar\theta^+\bar\chi^a-2i\theta^+\bar\theta^+\dot f^{ka}u^-_k\,.$$

- Free off-shell action:

$$S_{free}\sim\int dtd^4\theta du\, q^{+a}q^-_a\sim\int dt\left(\dot f^{ia}\dot f_{ia}-\frac{i}{2}\bar\chi^a\dot\chi_a\right),\quad q^{-a}:=D^{--}q^{+a}$$

- Nonlinear $d=1$ sigma model action:

$$S_{free}\sim\int dtd^4\theta du\,\mathcal{L}(q^{+a},q^{-b},u^\pm).$$

- In bosonic sector: HKT (“Hyper-Kähler with torsion”) sigma model. In components, the torsion appears in a term quartic in fermions.

How to construct general HK $\mathcal{N}=4, d=1$ sigma models? No torsion in this case, the geometry involves only Riemann curvature tensor. The answer was given in Delduc, I., 2010.

- The basic superfields are still real analytic, $q^{+A}(\zeta, u) = f^{iA}u_i^+ + ..., i = 1, 2,\ A = 1, \ldots 2n$, it encompasses just $4n$ fields $f^{iA}(t)$ parametrizing the target bosonic manifold, $\widetilde{(q_A^+)} = \Omega^{AB}q_B^+$, with $\Omega^{AB} = -\Omega^{BA}$ a constant symplectic metric.

- The linear constraint $D^{++}q^{+A} = 0$ is promoted to a nonlinear one
$$D^{++}q^{+A} = \Omega^{AB}\,\frac{\partial L^{+4}(q^{+C}, u^{\pm})}{\partial q^{+B}}.$$

- The superfield action is bilinear as in the free case,
$$S_{HK} \sim \int dt d^4\theta du\, \Omega^{AB} q_B^+ q_A^- = \int dt \left[g_{iA\,kB}(f)\dot{f}^{iA}\dot{f}^{kB} + \ldots\right],$$
the whole interaction appears only on account of nonlinear deformation of the q^{+A}-constraint.

- L^{+4} is an analytic hyper-Kähler potential (Galperin, I., Ogivetsky, Sokatchev, 1986): every L^{+4} produces the component HK metric $g_{iA\,kB}(f)$ and, *vice versa*, each HK metric originates from some HK potential L^{+4}.

From $\mathcal{N} = 4$ HK SQM to its QK deformation

The harmonic superspace approach supplies the most natural arena for defining $\mathcal{N} = 4$ QK SQM. Basic new features of these models as compared to their HK prototypes are as follows.

1. QK SQM model corresponding to $4n$ dimensional QK manifold requires introducing $n + 1$ multiplets $(4, 4, 0)$ described by analytic superfields $q^{+a}(\zeta, w^{\pm}), (a = 1, 2), Q^{+r}(\zeta, w^{\pm}), (r = 1, \ldots 2n)$. An extra superfield $q^{+a}(\zeta, w^{\pm})$ is $d = 1$ analog of $\mathcal{N} = 2, d = 4$ "conformal compensator".

2. QK SQM actions are invariant under local $\mathcal{N} = 4, d = 1$ supersymmetry realized by the appropriate transformations of super coordinates, including harmonic variables $w_i^{\pm}$.

3. For ensuring local invariance it is necessary to introduce a supervielbein $E(\zeta, \theta^-, \bar{\theta}^-, w^{\pm})$ which is a general $\mathcal{N} = 4, d = 1$ superfield.

4. Besides the (q^+, Q^+) superfield part, the correct action should contain a "comological term" depending on the vielbein superfield only.

Minimal local $\mathcal{N} = 4, d = 1$ SUSY

By analogy with the $\mathcal{N} = 2, d = 4$ case we postulate that local $\mathcal{N} = 4, d = 1$ SUSY preserves the Grassmann analyticity,

$$\begin{aligned}
&\delta t = \Lambda(\zeta, w)\,,\ \delta\theta^{+} = \Lambda^{+}(\zeta, w)\,,\ \delta\bar\theta^{+} = \bar\Lambda^{+}(\zeta, w)\,,\\
&\delta w_i^{+} = \Lambda^{++}(\zeta, w) w_i^{-}\ ,\ \delta w_i^{-} = 0\,,\\
&\delta\theta^{-} = \Lambda^{-}(\zeta, w, \theta^{-}, \bar\theta^{-})\,,\quad \delta\bar\theta^{-} = \bar\Lambda^{-}(\zeta, w, \theta^{-}, \bar\theta^{-})\,,
\end{aligned}$$

The explicit structure of the minimal set of analytic parameters is as follows

$$\begin{aligned}
\Lambda &= 2b + 2i(\lambda^i w_i^{-}\bar\theta^{+} - \bar\lambda^i w_i^{-}\theta^{+}) + 2i\theta^{+}\bar\theta^{+}\tau^{(ik)} w_i^{-} w_k^{-}\ ,\\
\Lambda^{+} &= \lambda^i w_i^{+} + \theta^{+}[\dot b + \tau^{(ik)} w_i^{+} w_k^{-}]\,,\\
\Lambda^{++} &= \tau^{(ik)} w_i^{+} w_k^{+} - 2i(\dot\lambda^i w_i^{+}\bar\theta^{+} - \dot{\bar\lambda}^i w_i^{+}\theta^{+}) - 2i\theta^{+}\bar\theta^{+}[\ddot b + \dot\tau^{(ik)} w_i^{+} w_k^{-}]\,,\\
\Lambda^{-} &= \lambda^i w_i^{-} + \theta^{+}\tau^{(ik)} w_i^{-} w_k^{-} + \theta^{-}[\dot b - \tau^{(ik)} w_i^{-} w_k^{+}]\\
&\quad - 2i\theta^{-}(\bar\theta^{+}\dot\lambda^i w_i^{-} - \theta^{+}\dot{\bar\lambda}^i w_i^{-}) + 2i\theta^{+}\bar\theta^{+}\theta^{-}\dot\tau^{(ik)} w_i^{-} w_k^{-}\ .
\end{aligned}$$

Here, $b(t)$, $\tau^{(ik)}(t)$ and $\lambda^i(t)$, $\bar\lambda^i(t)$ are arbitrary local parameters, bosonic and fermionic, respectively. The local $\mathcal{N} = 4, d = 1$ supergroup obtained is isomorphic to the classical (having no central charges) "small" $\mathcal{N} = 4$ superconformal symmetry.

How to generalize the $(4,4,0)$ superfields $q^{+A}(\zeta, w)$ to local SUSY?

- The simplest possibility is to still keep the linear constraint

$$D^{++}q^{+a} = 0\,.$$

However, because of the transformation law $\delta D^{++} = -\Lambda^{++}D^0$, with $D^0 q^{+a} = q^{+a}$, this constraint is covariant only if q^{+a} properly transforms.

- Observing that $\Lambda^{++} = D^{++}\Lambda_0$,

$$\Lambda_0 = \tau^{(ik)}w_i^+ w_k^- - \dot{b} + 2i(\bar{\theta}^+\dot{\lambda}^i - \theta^+\dot{\bar{\lambda}}^i)w_i^- - 2i\theta^+\bar{\theta}^+\dot{\tau}^{(ik)}w_i^- w_k^-\,,$$

it is easy to determine such a transformation law

$$\delta q^{+a} = \Lambda_0\, q^{+a}.$$

- To construct invariant actions, one needs the transformations of the integration measures $\mu_H := dtdwd^2\theta^+ d^2\theta^-$, $\mu^{(-2)} := dtdwd^2\theta^+$,

$$\delta\mu^{(-2)} = 0\,, \quad \delta\mu_H = \mu_H\, 2\Lambda_0\,,$$

and that of harmonic derivative D^{--},

$$\delta D^{--} = -(D^{--}\Lambda^{++})D^{--}, \quad D^{--}\Lambda^{++} = D^{++}\Lambda^{--},\ \Lambda^{--} := D^{--}\Lambda_0\,.$$

Simplest invariant action

Introduce, besides $q^{+a}(\zeta, w)$, $a = 1, 2,$, also extra superfields $Q^{+r}(\zeta, w)$, $r = 1, 2, \ldots 2n$, which encompass n off-shell multiplets $(4, 4, 0)$, obey the same linear harmonic constraint $D^{++}Q^{+r} = 0$ and transform under local $\mathcal{N} = 4$ SUSY in the same way as q^{+a} . The basic part of the total invariant action can be then written as

$$S_{(2)} = \int \mu_H \, E \, \mathcal{L}_{(2)}(q, Q) \,, \quad \mathcal{L}_{(2)}(q, Q) = \gamma q^{+a} q_a^- - Q^{+r} Q_r^- \,,$$

$$q_a^- := D^{--} q_a^+ \,, \quad Q_r^- := D^{--} Q_r^+ \,,$$

and $\gamma = \pm 1$. The new object is vielbein E which is harmonic-independent, $D^{++}E = D^{--}E = 0$, and possesses the following transformation law under local $\mathcal{N} = 4$ SUSY

$$\delta E = (-4\Lambda_0 + 2D^{--}\Lambda^{++})E \,.$$

The weight term is picked up so that $D^{++}(-4\Lambda_0 + 2D^{--}\Lambda^{++}) = 0$. This is not the end! One more important term should be added to $S_{(2)}$:

$$S_\beta = \beta \int \mu_H \sqrt{E} \,, \quad \delta S_\beta = \beta \int \mu_H \, D^{--}\Lambda^{++} \sqrt{E} = 0 \,.$$

Thus the simplest locally $\mathcal{N} = 4$ supersymmetric action reads

$$S_{HP} \sim S_{(2)} + S_\beta = \int \mu_H \big[E\,\mathcal{L}_{(2)} + \beta\sqrt{E}\big].$$

Why should the "cosmological constant" term S_β be added?

To answer this question, pass to the bosonic limit:

$$q^{+a} \;\Rightarrow\; f^{ia} w_i^+ - 2i\theta^+\bar\theta^+ \dot f^{ia} w_i^- \,, \quad Q^{+r} \;\Rightarrow\; F^{ir} w_i^+ - 2i\theta^+\bar\theta^+ \dot F^{ir} w_i^- \,,$$

$$\begin{aligned} E \;\Rightarrow\; & e + \theta^+\theta^- M - \bar\theta^+\bar\theta^- \bar M + \theta^+\bar\theta^-(\mu - i\dot e) + \bar\theta^+\theta^-(\mu + i\dot e) \\ & + 4i(\theta^+\bar\theta^+ w_i^- w_k^- - \theta^+\bar\theta^- w_i^- w_k^+ - \theta^-\bar\theta^+ w_i^- w_k^+ + \theta^-\bar\theta^- w_i^+ w_k^+) L^{(ik)} \\ & + 4\theta^+\bar\theta^+\theta^-\bar\theta^- [D + 2\dot L^{(ik)} w_i^+ w_k^-]. \end{aligned}$$

- In the bosonic limit,

$$
\begin{aligned}
L_{HP} \;\Rightarrow\; & \frac{1}{2} e \left(\dot{F}^{ir} \dot{F}_{ir} - \gamma \, \dot{f}^{ia} \dot{f}_{ia} \right) + L_{ik} \left[F^{(ir} \dot{F}_r^{k)} - \gamma f^{(ia} \dot{f}_a^{k)} \right] \\
& + \frac{1}{4} D \Big(\gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \beta \frac{1}{\sqrt{e}} \Big) \\
& + \frac{\beta}{4} \frac{1}{e^{3/2}} \left[L^{ik} L_{ik} - \frac{1}{8} \left(M\bar{M} + \mu^2 + \dot{e}^2 \right) \right].
\end{aligned}
$$

- The auxiliary fields M, $\bar{M}$ and μ fully decouple and can be put equal to zero by their equations of motion. Also, $e(t)$ is an analog of $d = 1$ vierbein, so it is natural to choose the gauge $e = 1$ ..
- Then the bosonic Lagrangian becomes

$$
\begin{aligned}
L_{HP} \;\Rightarrow\; & \frac{1}{2} \left(\dot{F}^{ir} \dot{F}_{ir} - \gamma \, \dot{f}^{ia} \dot{f}_{ia} \right) + L_{ik} \left[F^{(ir} \dot{F}_r^{k)} - \gamma f^{(ia} \dot{f}_a^{k)} \right] \\
& + \frac{1}{4} D \Big(\gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \beta \Big) + \frac{\beta}{4} L^{ik} L_{ik}.
\end{aligned}
$$

- At $\beta \neq 0$ L^{ik} can be eliminated by its algebraic equation of motion, while D serves as the Lagrange multiplier giving rise to the constraint relating f^{ia} and F^{ir}:

$$L^{ik} = -2\frac{1}{\beta}\left[F^{(ir}\dot{F}_r^{k)} - \gamma f^{(ia}\dot{f}_a^{k)}\right], \quad \gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \beta = 0\,.$$

- Assuming that f^{ia} starts with a constant (compensator!), one uses local $SU(2)$ freedom, $\delta f^{ia} = \tau^i{}_l f^{la}$, to gauge away the triplet from f^{ia},

$$f^{(ia)} = 0 \;\rightarrow\; f_a^i = \sqrt{2}\,\delta_a^i\,\omega\,.$$

- Then the constraint can be solved as

$$(a)\,\gamma = 1 \;\Rightarrow\; \beta < 0\,, \quad \omega = \frac{|\beta|^{1/2}}{2}\sqrt{1 + \frac{1}{|\beta|}F^2}\,,$$

$$(b)\,\gamma = -1 \;\Rightarrow\; \beta > 0\,, \quad \omega = \frac{\beta^{1/2}}{2}\sqrt{1 - \frac{1}{\beta}F^2}\,.$$

- The final form of the bosonic action for $\gamma = 1$ is

$$L_{HP} = \frac{1}{2}\Big[(\dot{F}\dot{F}) + \frac{2}{|\beta|}\,(F_{r(i}\dot{F}_{j)}{}^{r})(F_{s}^{(i}\,\dot{F}^{\,sj)}) - \frac{1}{|\beta|}\;\frac{1}{1+\frac{1}{|\beta|}\,F^{2}}(F\dot{F})(F\dot{F})\Big].$$

The option $\gamma = -1$ is recovered by the replacement $|\beta| \rightarrow -|\beta|$.

- These actions describe $d = 1$ nonlinear sigma models on non-compact and compact maximally "flat" $4n$ dimensional QK manifolds, respectively:

$$\widetilde{\mathbb{HP}}^{n} = \frac{Sp(1,n)}{Sp(1)\times Sp(n)}\,, \quad \mathbb{HP}^{n} = \frac{Sp(1+n)}{Sp(1)\times Sp(n)}\,.$$

- Thus $\mathcal{N} = 4$ mechanics constructed is just superextensions of these QK $d = 1$ sigma models.

Generalizations

- The basic step in generalizing to $\mathcal{N}=4$ mechanics with an arbitrary QK manifold is passing to nonlinear harmonic constraints

$$D^{++}q^{+a} - \gamma\frac{1}{2}\frac{\partial}{\partial q^{+}_{a}}\left[\hat{\kappa}^2(w^{-}\cdot q^{+})^2\mathcal{L}^{+4}\right] = 0\,,$$
$$D^{++}Q^{+r} + \frac{1}{2}\frac{\partial}{\partial Q^{+}_{r}}\left[\hat{\kappa}^2(w^{-}\cdot q^{+})^2\mathcal{L}^{+4}\right] = 0\,,$$
$$\mathcal{L}^{+4} \equiv \mathcal{L}^{+4}\Big(\frac{Q^{+r}}{\hat{\kappa}(w^{-}q^{+})}, \frac{q^{+a}}{(w^{-}q^{+})}, w^{-}_{i}\Big), \quad \hat{\kappa} := \frac{\sqrt{2}}{|\beta|^{1/2}}.$$

- The invariant superfield action looks the same as in the $\mathbb{HP}^n$ case

$$\mathbb{S}_{QK} \sim \left[\tilde{S}_{(2)} + S_{\beta}\right] = \int \mu_H\Big[E\,\tilde{\mathcal{L}}_{(2)} + \beta\sqrt{E}\Big],$$
$$\tilde{\mathcal{L}}_{(2)} = \gamma q^{+a}q^{-}_{a} - Q^{+r}Q^{-}_{r}\,, \quad q^{-}_{a} = D^{--}q^{+a}\,, \quad Q^{-}_{r} = D^{--}Q^{+r}\,.$$

- The bosonic action precisely coincides with $d=1$ reduction of the general QK sigma model action derived from $\mathcal{N}=2, d=4$ supergravity-matter action in E.I., G. Valent, 2000. This coincidence proves that we have constructed most general QK $\mathcal{N}=4$ mechanics.

- One more possibility is to consider the following generalization of the $\mathbb{HP}^n$ action

$$S^{loc}(q,Q) = \int \mu_H \sqrt{E}\,\mathcal{F}(X,Y,w^-),\ X := \sqrt{E}\,(q^{+a}q^-_a)\,,\ Y := \sqrt{E}\,(Q^{+r}Q^-_r),$$
$$D^{++}q^{+a} = D^{++}Q^{+r} = 0 \quad \Rightarrow \quad D^{\pm\pm}X = D^{\pm\pm}Y = 0\,.$$

- When $E = const$, it is reduced to the particular form of the HKT action $\int \mu_H \mathcal{F}(q^{+A}, q^{-B}, w^{\pm})$, while for $\mathcal{F}(X,Y,w^-) = \gamma\, X - Y + \beta$ just to $\mathbb{HP}^n$ action. So the target geometry associated with $S^{loc}(q,Q)$ is expected to be a kind of QKT, i.e. “Quaternion-Kähler with torsion”. To date, not too much known about such geometries...

Summary and Outlook

- Two different deformations of $\mathcal{N} = 8$ supersymmetric mechanics based on the supergroups $SU(2|2)$ and $SU(4|1)$ as a generalization of the $SU(2|1)$ mechanics were sketched.

- $\mathcal{N} = 4, d = 1$ harmonic superspace methods were used to construct a new class of $\mathcal{N} = 4$ supersymmetric mechanics models, those with $d = 1$ Quaternion-Kähler sigma models as a bosonic core. The basic distinguishing feature of these models is *local* $\mathcal{N} = 4, d = 1$ supersymmetry.

- The superfield and component actions were presented for general $\mathcal{N} = 4$ QK mechanics, the simplest case of the maximally "flat" $\mathbb{HP}^n$ mechanics was considered in detail.

- A few generalizations of QK mechanics were proposed, in particular "Quaternion-Kähler with torsion" (QKT) models.

- *Some further lines of study*:

  (a) To construct the Hamiltonian formalism for the new class of mechanical systems, including $\mathcal{N} = 4$ supercharges. To perform quantization, at least for the simplest case of $\mathbb{HP}^n$ mechanics, to find the energy spectrum.

  (b) To explicitly construct some other particular $\mathcal{N} = 4$ QK SQM models, e.g. associated with symmetric QK manifolds ("Wolf spaces").

  (c) In particular, to construct locally supersymmetric versions of other off-shell $\mathcal{N} = 4, d = 1$ multiplets (such as $(3, 4, 1)$, $(1, 4, 3)$, etc) and the associated SQM systems (Landau-type, Calogero-Moser-type and others).

  (d) Links between the two types of SQM deformations presented in this talk?

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