The asymptotic approach to renormalization of the Yang-Mills theory

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Yang–Mills theory

The basic concepts of this work

Assume, $G$ is a compact group of charges, $\mathfrak{G}$ is the Lie algebra. Assume $t^a$ are the generators of the Lie algebra. $\text{tr}[\cdot, \cdot]$ is the Killing form, $d = 4$

$$A(x) = A^a_\mu(x) t^a dx^\mu$$

$$F = dA + A \wedge A$$

The classical action of the theory of Yang–Mills

$$S = \frac{1}{4g^2} \int \text{tr} \ F \wedge F^*,$$

where $g = \frac{\sqrt{\alpha}}{2}$, $\alpha$ is the coupling constant.
Effective action

The differential operators defining the quadratic form

\[ M_1 = \nabla^2_\otimes \delta_{\mu\nu} + 2[F_{\mu\nu}, \cdot] , \quad M_0 = \nabla^2_\mu . \]

The action is expanded in a series of views

\[ W(B, \alpha) = \frac{1}{\alpha} W_{-1}(B) + \sum_{k=0}^{\infty} \alpha^k W_k(B) , \]

where

\[ W_{-1}(B) = \int \text{tr} F_{\mu\nu}^2 d^4 x \quad W_0(B) = -\frac{1}{2} \ln \det M_1 + \ln \det M_0 \]

and \( W_n, n = 1, 2 \ldots \) are defined as the contribution of strongly connected vacuum diagrams with \( n + 1 \) loops, constructed via Green functions.
Z-function

Renormalization

\[ \alpha \mapsto \alpha(\varepsilon) \implies \alpha_r = \alpha_r(\mu, \alpha(\varepsilon), \varepsilon), \]

where the relation exists

\[ \alpha(\varepsilon) = \left( \frac{\mu}{m} \right)^\varepsilon \alpha_r Z(\alpha_r, \varepsilon). \]

The standard form of \( Z(\alpha_r, \varepsilon) \) is

\[ Z(\alpha_r, \varepsilon) = 1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{z_{n,k} \alpha_r^n}{\varepsilon^k} \xrightarrow{\varepsilon \to +0} ??? \]

The answer: using of Gell-Mann-Low equation for coupling constant \( \alpha_r \).
The Gell-Mann-Low problem

\[
\begin{align*}
\mu \frac{d}{d\mu} \alpha_r &= -\varepsilon \alpha_r + \sum_{i=0}^{\infty} \beta_{i+1} \alpha_r^{i+2} = -\varepsilon \alpha_r + \beta(\alpha_r); \\
\alpha_r \big|_{\mu=1} &= \theta(1, \varepsilon).
\end{align*}
\]

\[
\downarrow \quad \frac{d}{d\mu} \bigg|_{\mu=m} \alpha(\varepsilon) = \left( \frac{\mu}{m} \right)^{\varepsilon} \alpha_r Z(\alpha_r, \varepsilon)
\]

\[
\begin{align*}
Z'_\alpha (\alpha_r, \varepsilon) - \frac{\beta(x)dx}{x(\varepsilon x - \beta(x))} Z(\alpha_r, \varepsilon) &= 0; \\
Z(0, \varepsilon) &= 1.
\end{align*}
\]
The classical solution for Z-function

\[ Z(\alpha_r, \varepsilon) = \exp \left( \int_0^{\alpha_r} \frac{\beta(x) \, dx}{x(\varepsilon x - \beta(x))} \right) . \]

\[ \alpha(\varepsilon) = \left( \frac{\mu}{m} \right)^\varepsilon \alpha_r \exp \left( \int_0^{\alpha_r} \frac{\beta(x) \, dx}{x(\varepsilon x - \beta(x))} \right) . \]
Asymptotic of $\alpha(\varepsilon)$

When $\varepsilon \to +0$ then

$$Z(\alpha_r, \varepsilon) = \left( \frac{\varepsilon}{\varepsilon - \beta_1 \alpha_r} \right) (1 + o(1)).$$

- if $\alpha_r$ runs to zero slower then $\varepsilon$ or runs to nonzero constant hence

$$\alpha(\varepsilon) = -\frac{\varepsilon}{\beta_1} (1 + o(1));$$

- if $\alpha_r = a\varepsilon(1 + o(1))$ and $a \neq \beta_1^{-1}$ therefore

$$\alpha(\varepsilon) = \frac{a\varepsilon}{1 - \beta_1 a} (1 + o(1));$$

- if $\alpha_r$ runs to zero faster then $\varepsilon$ therefore

$$\alpha(\varepsilon) = \alpha_r (1 + o(1)).$$
Asymptotics

Asymptotic of $\alpha(\varepsilon)$

If $\alpha_r$ goes to nonzero constant for $\varepsilon \to +0$ then

$$\alpha(\varepsilon) = -\frac{\varepsilon}{\beta_1} (1 + o(1)),$$

$$\frac{1}{\alpha(\varepsilon)} = -\frac{\beta_1}{\varepsilon} + c_1 \ln(\varepsilon) + C + o(1),$$

where

$$C = \ln\left(\frac{\mu}{m}\right) + \frac{1}{\beta_1 \alpha} + \frac{\beta_2}{\beta_1^2} \ln\left(\frac{\beta_1^2 \alpha}{-\beta_1 - \beta_2 \alpha}\right) + \int_0^\alpha \frac{f(x)dx}{\beta(x)g(x)},$$

and

$$\mu \frac{dC}{d\mu} = 0.$$
The first correction

Effective action before regularization

\[ W(B, \alpha) = \frac{1}{\alpha} W_{-1}(B) + W_0(B) + \alpha W_1(B) + \alpha^2 W_2(B) + \ldots \]

\[ W_0(B) = \int_0^\infty \frac{\tau d\tau}{\tau^{d/2}} T(\tau), \]

where \( T(\tau) \) is the trace of heat kernel. After regularization

\[ W_0^{\text{reg}}(B) = \frac{1}{\varepsilon} p^{-\varepsilon} \beta_1 W_{-1}(\varepsilon) + W_{0,0}(\varepsilon), \]

where \( p \) - auxiliary massive parameter.
The other corrections

For $k > 0$ regularization has the form

$$W_k \mapsto \mu^k \epsilon \left( \sum_{i=1}^{k} \frac{1}{\epsilon_i} W_{k,i} + W_{k,0}(\epsilon) \right).$$

So effective action after regularization has the form

$$W_{\text{reg}}(B) = \frac{m^{-\epsilon}}{\alpha \left( \frac{\mu}{m} \right)^\epsilon} W_{-1}(\epsilon) + \frac{1}{\epsilon} W_{0,1} + W_{0,0} +$$

$$+ \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{1}{\epsilon_j} W_{k,j} \left( \alpha \left( \frac{\mu}{m} \right)^\epsilon \right)^k + \sum_{k=1}^{\infty} W_{k,0}(\epsilon) \left( \alpha \left( \frac{\mu}{m} \right)^\epsilon \right)^k.$$
\[ W_{\text{reg}}(B) = \frac{m^{-\varepsilon}}{\alpha \left( \frac{\mu}{m} \right) \varepsilon} W_{-1}(\varepsilon) + \frac{1}{\varepsilon} W_{0,1} + W_{0,0}(\varepsilon) + \]

\[ + \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{1}{\varepsilon^j} W_{k,j} \left( \alpha \left( \frac{\mu}{m} \right) \varepsilon \right)^k + \sum_{k=1}^{\infty} W_{k,0}(\varepsilon) \left( \alpha \left( \frac{\mu}{m} \right) \varepsilon \right)^k. \]

After using asymptotic for coupling constant for \( \varepsilon \to +0 \)

\[ \frac{m^{-\varepsilon}}{\alpha(\varepsilon) \left( \frac{\mu}{m} \right) \varepsilon} W_{-1}(\varepsilon) + \frac{1}{\varepsilon} W_{0,1} \overset{\text{\( \rightarrow \)}}{=} \left( C + \beta_1 \ln(\mu/p) \right) W_{-1}. \]

\[ W_{0,0}(\varepsilon) \overset{\text{\( \rightarrow \)}}{=} W_{1 \text{ loop}}(p). \]

\[ \left( \alpha(\varepsilon) \left( \frac{\mu}{m} \right) \varepsilon \right)^k \left( \sum_{i=1}^{k} \frac{1}{\varepsilon^i} W_{k,i} + W_{k,0}(\varepsilon) \right) \overset{\text{\( \rightarrow \)}}{=} (-1)^k \frac{W_{k,k}}{\beta_1^k}, \quad k > 0, \]
After substitution $p = \mu$ the formula for the effective action has the form

$$W_{\text{ren}}(B) = CW_{-1} + W_{1\,\text{loop}}(\mu) + \sum_{k=1}^{\infty} (-1)^k \frac{W_{k,k}}{\beta_1^k}.$$ 

Many thanks