The asymptotic approach to renormalization of the Yang-Mills theory

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Yang–Mills theory

The basic concepts of this work

Assume, G is a compact group of charges, $\mathfrak{G}$ is the Lie algebra.
Assume t^a are the generators of the Lie algebra.
$\mathrm{tr}[\cdot,\cdot]$ is the Killing form, $d = 4$

$$A(x) = A^a_\mu(x) t^a dx^\mu$$

$$F = dA + A \wedge A$$

The classical action of the theory of Yang — Mills

$$S = \frac{1}{4g^2} \int \mathrm{tr}\, F \wedge F^*,$$

where $g = \frac{\sqrt{\alpha}}{2}$, a α is the coupling constant.

Effective action

The differential operators defining the quadratic form

$$M_1 = \nabla^2_\sigma \delta_{\mu\nu} + 2[F_{\mu\nu}, \cdot\,]\,, \qquad M_0 = \nabla^2_\mu .$$

The action is expanded in a series of views

$$W(B,\alpha) = \frac{1}{\alpha} W_{-1}(B) + \sum_{k=0}^{\infty} \alpha^k W_k(B)\,,$$

where

$$W_{-1}(B) = \int \mathrm{tr} F^2_{\mu\nu} d^4x \qquad W_0(B) = -\tfrac{1}{2}\ln\det M_1 + \ln\det M_0$$

and W_n, $n = 1, 2...$ are defined as the contribution of strongly connected vacuum diagrams with $n+1$ loops, constructed via Green functions.

Z-function

Renormalization

$$\alpha \longmapsto \alpha(\varepsilon) \Longrightarrow \alpha_r = \alpha_r(\mu, \alpha(\varepsilon), \varepsilon)\,,$$

where the relation exists

$$\alpha(\varepsilon) = \left(\frac{\mu}{m}\right)^{\varepsilon} \alpha_r Z(\alpha_r, \varepsilon)\,.$$

The standart form of $Z(\alpha_r, \varepsilon)$ is

$$Z(\alpha_r, \varepsilon) = 1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{z_{n,k}\alpha_r^n}{\varepsilon^k} \xrightarrow{\varepsilon \to +0} ???$$

The answer: using of Gell-Mann-Low equation for coupling constant α_r.

The Gell-Mann-Low problem

$$\begin{cases} \mu\frac{d}{d\mu}\alpha_r = -\varepsilon\alpha_r + \sum\limits_{i=0}^{\infty}\beta_{i+1}\alpha_r^{i+2} = -\varepsilon\alpha_r + \beta(\alpha_r); \\ \alpha_r\big|_{\mu=1} = \theta(1,\varepsilon). \end{cases}$$

$$\Downarrow \; \frac{d}{d\mu}\Big| \quad \alpha(\varepsilon) = \left(\frac{\mu}{m}\right)^{\varepsilon}\alpha_r Z(\alpha_r,\varepsilon)$$

$$\begin{cases} Z'_{\alpha_r}(\alpha_r,\epsilon) - \frac{\beta(x)dx}{x(\varepsilon x-\beta(x))}Z(\alpha_r,\epsilon) = 0; \\ Z(0,\epsilon) = 1. \end{cases}$$

The classical solution for Z-function

$$Z(\alpha_r,\varepsilon)=\exp\left(\int\limits_0^{\alpha_r}\frac{\beta(x)dx}{x(\varepsilon x-\beta(x))}\right).$$

$$\Downarrow$$

$$\alpha(\varepsilon)=\left(\frac{\mu}{m}\right)^{\varepsilon}\alpha_r\exp\left(\int\limits_0^{\alpha_r}\frac{\beta(x)dx}{x(\varepsilon x-\beta(x))}\right).$$

Asymptotic of $\alpha(\varepsilon)$

When $\varepsilon \to +0$ then

$$Z(\alpha_r, \varepsilon) = \left(\frac{\varepsilon}{\varepsilon - \beta_1 \alpha_r}\right)(1 + o(1)).$$

- if α_r runs to zero slower then ε or runs to nonzero constant hense

$$\alpha(\varepsilon) = -\frac{\varepsilon}{\beta_1}(1 + o(1));$$

- if $\alpha_r = a\varepsilon(1 + o(1))$ and $a \neq \beta_1^{-1}$ therefore

$$\alpha(\varepsilon) = \frac{a\varepsilon}{1 - \beta_1 a}(1 + o(1));$$

- if α_r runs to zero faster then ε therefore

$$\alpha(\varepsilon) = \alpha_r(1 + o(1)).$$

Asymptotic of $\alpha(\varepsilon)$

If α_r goes to nonzero constant for $\varepsilon \to +0$ then

$$\alpha(\varepsilon) = -\frac{\varepsilon}{\beta_1}(1 + o(1)),$$

$$\frac{1}{\alpha(\varepsilon)} = -\frac{\beta_1}{\varepsilon} + c_1 \ln(\varepsilon) + C + o(1),$$

where

$$C = \ln\left(\frac{\mu}{m}\right) + \frac{1}{\beta_1 \alpha} + \frac{\beta_2}{\beta_1^2} \ln\left(\frac{\beta_1^2 \alpha}{-\beta_1 - \beta_2 \alpha}\right) + \int_0^{\alpha} \frac{f(x)dx}{\beta(x)g(x)}$$

and

$$\mu \frac{dC}{d\mu} = 0 .$$

The first correction

Effective action before regularization

$$W(B,\alpha)=\frac{1}{\alpha}W_{-1}(B)+W_0(B)+\alpha W_1(B)+\alpha^2 W_2(B)+\ldots$$

$$W_0(B)=\int\limits_0^\infty \frac{\tau d\tau}{\tau^{d/2}}T(\tau)\,,$$

where $T(\tau)$ is the trace of heat kernal. After regularization

$$W_0^{reg}(B)=\frac{1}{\varepsilon}p^{-\varepsilon}\beta_1 W_{-1}(\varepsilon)+W_{0,0}(\varepsilon)\,,$$

where p - auxiliary massive parameter.

The other corrections

For $k > 0$ regularization has the form

$$W_k \longmapsto \mu^{k\varepsilon}\left(\sum_{i=1}^{k}\frac{1}{\varepsilon^i}W_{k,i} + W_{k,0}(\varepsilon)\right).$$

So effective action after regularization has the form

$$W_{reg}(B) = \frac{m^{-\varepsilon}}{\alpha\left(\frac{\mu}{m}\right)^{\varepsilon}}W_{-1}(\varepsilon) + \frac{1}{\varepsilon}W_{0,1} + W_{0,0} +$$
$$+\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\frac{1}{\varepsilon^j}W_{k,j}\left(\alpha\left(\frac{\mu}{m}\right)^{\varepsilon}\right)^k + \sum_{k=1}^{\infty}W_{k,0}(\varepsilon)\left(\alpha\left(\frac{\mu}{m}\right)^{\varepsilon}\right)^k.$$

$$W_{reg}(B) = \frac{m^{-\varepsilon}}{\alpha\left(\frac{\mu}{m}\right)^{\varepsilon}} W_{-1}(\varepsilon) + \frac{1}{\varepsilon} W_{0,1} + W_{0,0}(\varepsilon) +$$

$$+ \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{1}{\varepsilon^{j}} W_{k,j} \left(\alpha\left(\frac{\mu}{m}\right)^{\varepsilon}\right)^{k} + \sum_{k=1}^{\infty} W_{k,0}(\varepsilon) \left(\alpha\left(\frac{\mu}{m}\right)^{\varepsilon}\right)^{k}.$$

After using asymptotic for coupling constant for $\varepsilon \to +0$

$$\frac{m^{-\varepsilon}}{\alpha(\varepsilon)\left(\frac{\mu}{m}\right)^{\varepsilon}} W_{-1}(\varepsilon) + \frac{1}{\varepsilon} W_{0,1} \longmapsto \left(C + \beta_1 \ln(\mu/p)\right) W_{-1}\,.$$

$$W_{0,0}(\varepsilon) \longmapsto W_{1\ loop}(p).$$

$$\left(\alpha(\varepsilon)\left(\frac{\mu}{m}\right)^{\varepsilon}\right)^{k} \left(\sum_{i=1}^{k} \frac{1}{\varepsilon^{i}} W_{k,i} + W_{k,0}(\varepsilon)\right) \longmapsto (-1)^{k} \frac{W_{k,k}}{\beta_1^{k}}\,,\ k > 0\,,$$

After substitution $p = \mu$ the formula for the effective action has the form

$$W_{ren}(B) = CW_{-1} + W_{1\,loop}(\mu) + \sum_{k=1}^{\infty}(-1)^k \frac{W_{k,k}}{\beta_1^k}\,.$$

A. V. Ivanov, “About dimensional regularization in the Yang–Mills theory”, Zap. Nauchn. Sem. POMI, 465, POMI, St. Petersburg, 2017, 147–156

Many thanks