

# Inhomogeneous perturbations in the pseudo-Hermitian quantum cosmology

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- Motivation - cosmology of phantom fields
- Pseudo-Hermitian quantum mechanics
- Application of pseudo-Hermitian Hamiltonians to the phantom models
- The WKB probability current in the ordinary and pseudo-Hermitian way
- Example with exact minisuperspace solutions

# Dark Energy

- According to observations after inflation the Universe was spatially-flat, homogeneous and isotropic with fine structure representing small fluctuations over a flat space,

$$ds^2 = d\tau^2 - a^2(\tau)d\vec{x}^2 \quad (1)$$

induced by a diagonal energy-momentum tensor

$$T_{\mu\nu} = \text{diag}(\epsilon, p, p, p) \quad (2)$$

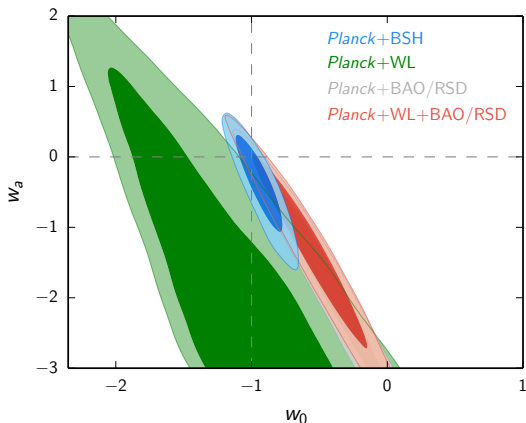
Hubble variable  $h \equiv \frac{\dot{a}}{a}$  characterizes the Universe expansion and satisfies the Friedmann equation

$$h^2 = \frac{\kappa^2}{3}\epsilon, \quad p = -\frac{2\dot{h}}{\kappa^2} - \frac{3h^2}{\kappa^2}, \quad \kappa^2 = 8\pi G = M_{Pl}^{-2} \quad (3)$$

- The average energy density  $\epsilon$  and pressure  $p$  are somewhat unusual: dark and visible matter are dominated by something called *dark energy* with equation of state  $w = p/\epsilon \sim -1$ .

# Dark Energy equation of state

The simplest explanation is cosmological constant with  $w = -1$  however the so-called *phantom matter* with  $w < -1$  has not been excluded. For ansatz  $w = w_0 + w_a(1 - a) + O((1 - a)^2)$  according to PLANCK, 1502.01590



# Phantom scalar matter

The phantom matter can be e.g. a scalar field  $\xi$  possessing negative kinetic energy,

$$L_{phantom} = -\frac{1}{2}\partial_\mu\xi\partial^\mu\xi - V(\tilde{\xi}) \quad (4)$$

However this leads to severe instabilities because its energy is not bounded from below and cosmological evolution may end up in the Big Rip. We propose to describe them with classically equivalent model,

$$L_{PTom} = \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} - V(i\tilde{\Phi}) \quad (5)$$

with  $PT$  symmetry  $\tilde{\Phi} \mapsto -\tilde{\Phi}$ ,  $i \mapsto -i$ . The perturbations should be considered along the real axis,

$$\tilde{\Phi} = i\xi_{class} + \delta\Phi \quad (6)$$

Such perturbations near classical trajectory happen to possess positively definite effective Hamiltonian. To separate the fields with  $PT$ -symmetry from usual phantoms we coin a new name -  $PTom$ . The aim of this work is to explore the possibility to construct quantum model of  $PToms$ .

# PT-symmetric non-Hermitian Hamiltonians

Consider the following Hamiltonian,

$$H = p^2 + x^2(ix)^\epsilon \quad (7)$$

While it's not Hermitian for  $\epsilon > 2$  it is symmetric under PT-symmetry,

$$\mathcal{PT} : x \mapsto -x, p \mapsto p \quad i \mapsto -i \quad (8)$$

It happens that for this Hamiltonian and many other PT-symmetric Hamiltonians all eigenvalues happen to be real and positive even for  $\epsilon = 2$  (Bender, Boettcher, 1997). Its eigenfunctions of course are not orthogonal in terms of the initial norm but rather some new norm,

$$\langle \phi_n | CPT | \phi_m \rangle = \delta_{nm}, \quad C^\dagger = -C, \quad [C, H] = 0, \quad [C, \mathcal{PT}] = 0 \quad (9)$$

From the point of view of this new norm the Hamiltonian can be considered Hermitian and generates unitary evolution.

# Pseudo-Hermitian quantum mechanics

PT-symmetric Hamiltonians are a particular case of the Pseudo-Hermitian quantum mechanics.

$$H = \eta^{-1} h \eta, \quad h = h^\dagger, \quad \eta^\dagger \eta \neq 1 \quad (10)$$

One may either use the Hermitian Hamiltonian  $h$  and the corresponding norm  $\langle \psi | \phi \rangle$  to compute probabilities or the equivalent description with non-Hermitian Hamiltonian  $H$  and new norm  $\langle \psi | \eta^\dagger \eta | \phi \rangle$ .

However e.g. for  $H = p^2 + x^2(ix)^\epsilon$  the similarity transformation  $\eta$  is known only in perturbation theory and the corresponding  $h$  is highly nonlocal.

**Thus naively non-Hermitian or unbounded from below Hamiltonian may describe unitary evolution of some stable but very complicated system.**

For time-dependent Hamiltonian  $h(t)$  and similarity operator  $\eta(t)$  the equivalent non-Hermitian Hamiltonian no longer is Pseudo-Hermitian and may have imaginary eigenvalues,

$$H = \eta^{-1} h \eta - i \eta^{-1} \dot{\eta} \quad (11)$$

## Quintessence + PTom model

In order to fit observations we need a composition of two scalar fields: quintessence and PTom ones. Let's consider the following model,

$$S = \int d^4x \sqrt{-g} \left( -\frac{1}{2\kappa^2} R + \frac{1}{2} M_{\Phi\Phi} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} M_{\tilde{\Phi}\tilde{\Phi}} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} + iM_{\Phi\tilde{\Phi}} \partial_\mu \Phi \partial^\mu \tilde{\Phi} - V(\Phi) + \tilde{V}(\tilde{\Phi}) \right), \quad (12)$$

where all parameters are real,  $V(\Phi)^* = V(\Phi)$  and  $\tilde{V}(\tilde{\Phi})^* = \tilde{V}(-\tilde{\Phi})$  to preserve the following symmetry,

$$t \mapsto -t, \quad i \mapsto -i, \quad \Phi \mapsto \Phi, \quad \tilde{\Phi} \mapsto -\tilde{\Phi}. \quad (13)$$

Easy to study example,

$$V = V_0 e^{\lambda\Phi}, \quad \tilde{V} = -\tilde{V}_0 e^{i\tilde{\lambda}\tilde{\Phi}} \quad (14)$$



# Longitudinal inhomogeneous modes

We will demonstrate our approach on the longitudinal (3d scalar) modes that decouple from transverse (tensor and vector) ones at quadratic order. The tensor and vector fluctuations are added in the same fashion,

$$g_{\mu\nu} = (N^2(t) + s(t, x))dt^2 + 2(\partial_k v(t, x))dt dx^k - e^{2\rho} \left( \delta_{ij} + h(t, x)\delta_{ij} + \partial_i \partial_j E(t, x) \right) dx^i dx^j \quad (15)$$

$$\Phi = \Phi(t) + \phi(t, x), \quad \tilde{\Phi} = \tilde{\Phi}(t) + \tilde{\phi}(t, x) \quad (16)$$

Let us choose the partial gauge  $h = E = 0$  and decompose into eigenfunctions of Laplace operator on the space with IR regulator  $L$ .

$$\phi(t, x) = \sum_{\Omega} \phi(t, \Omega) f(\Omega, x), \quad -\Delta f(\Omega, x) = \Omega^2 f(\Omega, x) \quad (17)$$

We get the system of constraints,

$$p_N = 0, \quad L^3 H_0 + \sum_{\Omega} H_2(\Omega) = 0, \quad p_s = 0, \quad p_v = 0 \quad (18)$$

Thus  $s$  and  $v$  are undynamical variables that can be get rid of after solving secondary constraints.

# Wheeler-DeWitt equation

Let us introduce the wavefunctional  $\Psi \left[ \rho, \Phi, \tilde{\Phi}, N \left| \{ \phi, \tilde{\phi}, s, v \} \right. \right]$  and perform the canonical quantization of the constraints,

$$\left[ L^3 \hat{H}_0 + \sum_{\Omega} \hat{H}_2(\Omega) \right] \Psi = 0, \quad \hat{\rho}_N \Psi = 0 \quad \hat{\rho}_s \Psi = 0, \quad \hat{\rho}_v \Psi = 0, \quad (19)$$

After quantization we assume Born-Oppenheimer approximation,

$$\Psi = \Psi_0(\rho, \Phi, \tilde{\Phi}) \psi_2[\phi, \tilde{\phi} | \rho, \Phi, \tilde{\Phi}] \quad (20)$$

$\Psi_0$  satisfies the minisuperspace WdW equation,  $\{\Phi_a\} = (\Phi, \tilde{\Phi})$

$$\left[ \frac{\kappa^2}{12L^6} \partial_{\rho}^2 - \frac{1}{2L^6} (M^{-1})_{ab} \partial_a^2 + \left( V(\Phi) - \tilde{V}(\tilde{\Phi}) \right) e^{6\rho} \right] \Psi_0 = 0 \quad (21)$$

# Fluctuation evolution

For large  $L$  let us assume that  $\Psi_0 \simeq \mathcal{A}e^{iL^3 S}$  then we get equation,

$$\frac{i\kappa^2}{6}(\partial_\rho S)(\partial_\rho \psi_2) - i(M^{-1})_{ab}(\partial_{\Phi_a} S)(\partial_{\Phi_b} \psi_2) = \sum_{\Omega} H_2(\Omega)\psi_2,$$

where indices  $a, b = \Phi, \tilde{\Phi}$ . The lhs is usually interpreted as derivative  $\frac{i}{L^3}\partial_\tau$  in so-called WKB-time that is taken along the classical trajectory,

$$\frac{i\kappa^2}{6}(\partial_\rho S)(\partial_\rho \tau) - i(M^{-1})_{ab}(\partial_{\Phi_a} S)(\partial_{\Phi_b} \tau) = 1 \quad (22)$$

The equation on  $\psi_2$  for each mode becomes the Schrödinger equation for the harmonic oscillator with time-dependent frequency in the WKB-time  $\eta$ . For small  $\eta$  this coincides with the evolution equation of the modes computed on the classical background.

In our case the frequencies are generally complex.

# Probability current

Because the WdW equation is of the Klein-Gordon type the problem arises of defining the appropriate inner product and probabilities. We will use a practical approach that works in the WKB region of  $\Psi_0$ .

Let us start with ordinary fields. Consider the WKB-wavepacket in the minisuperspace  $Q_A = (\rho, \Phi_a)$

$$\Psi_0 \simeq \mathcal{A} e^{iL^3 S - L^3 R} \quad (23)$$

In the center of this wavepacket  $|\partial_A S| \gg |\partial_A R|$ . We may introduce the conserved current,

$$J_A \simeq |\mathcal{A}|^2 \partial_A S, \quad \mathcal{G}^{AB} \partial_A J_B \simeq 0 \quad (24)$$

Different  $R$  may allow different  $\mathcal{A}$ . This way we can treat all wavepackets with similar  $S$  as having a probability amplitude  $\mathcal{A}$  defined on the surface of constant  $S$ .  $J_A$  integrated over regions in the center of the wavepackets allows to introduce conditional probabilities.

## Probability current 2

We choose new coordinates in the minisuperspace  $(\eta, Q_a^\perp)$  with  $Q^\perp$  parameterizing some surface of constant  $S$ .

We can combine the conservation of  $J_A$  in the minisuperspace with the evolution of  $\psi_2$  in  $\eta$  that admits a conserved inner product  $\langle \psi_2^{(1)} | \psi_2^{(2)} \rangle_{Q^\perp}$  for each classical trajectory moving from  $Q^\perp$ . Then for two wavepackets with the same  $S$  on the surface  $\Sigma$  of constant  $S$  we can define,

$$\langle \Psi^{(1)} | \Psi^{(2)} \rangle = \int_{\Sigma} d\Sigma \left[ \mathcal{A}^{(1)}(Q^\perp) \right]^* \mathcal{A}^{(2)}(Q^\perp) \langle \psi_2^{(1)} | \psi_2^{(2)} \rangle_{Q^\perp} \quad (25)$$

This allows us to define conditional probabilities for the full minisuperspace+fluctuations quantum cosmology

# Probability current P Tom

In pseudo-Hermitian case we can generalize this approach using the  $\mathcal{PT}$  symmetry when classical trajectories are  $\mathcal{PT}$ -symmetric (and therefore we can define  $\tau$  in  $\mathcal{PT}$ -symmetric fashion)

For fluctuation evolution we use the pseudo-Hermitian representation with time-dependent  $\eta$ ,

$$H_{2,\Omega}(\tau, Q^\perp) = \eta^{-1}(\tau, Q^\perp) h_{2,\Omega}(\tau, Q^\perp) \eta - i\eta^{-1}(\tau, Q^\perp) \partial_\tau \eta(\tau, Q^\perp) \quad (26)$$

The  $\mathcal{PT}$  symmetry under the minisuperspace variables implies the symmetry of  $\eta$ .

In the minisuperspace we can generalize the inner product density using  $\mathcal{PT}$  transformation on the surface of constant  $S$ .

$$\langle \Psi^{(1)} | \Psi^{(2)} \rangle = \int_\Sigma d\Sigma \left[ \mathcal{A}^{(1)}(\mathcal{PT}Q^\perp) \right] \mathcal{A}^{(2)}(Q^\perp) (\psi_2^{(1)} | \psi_2^{(2)})_{Q^\perp} \quad (27)$$

where,

$$(\psi_2^{(1)} | \psi_2^{(2)})_{Q^\perp} = \int_{\Sigma=\mathcal{PT}\Sigma} \prod_a dQ_a^\perp \psi_2^{(2)}(\tau, \mathcal{PT}Q^\perp) \eta^\dagger(\tau, \mathcal{PT}Q^\perp) \eta(\tau, Q^\perp) \psi_2^{(1)}(Q^\perp) \quad (28)$$

# Integrable minisuperspace model example

For the exponential potential with special form of kinetic matrix,

$$\lambda \tilde{\lambda} \frac{M_{\Phi\tilde{\Phi}}}{\mathcal{D}} = 6\kappa^2, \quad D = \lambda^2 \frac{M_{\tilde{\Phi}\tilde{\Phi}}}{\mathcal{D}} - 6\kappa^2, \quad \tilde{D} = \tilde{\lambda}^2 \frac{M_{\Phi\Phi}}{\mathcal{D}} + 6\kappa^2 \quad (29)$$

where  $\mathcal{D} = M_{\Phi\Phi}M_{\tilde{\Phi}\tilde{\Phi}} + M_{\Phi\tilde{\Phi}}^2$ , after the following transformation:

$$\chi = \lambda\Phi + 6\rho, \quad \pi = \frac{1}{\lambda}p_\Phi, \quad \tilde{\chi} = \tilde{\lambda}\tilde{\Phi} - 6i\rho, \quad \tilde{\pi} = \frac{1}{\tilde{\lambda}}p_{\tilde{\Phi}}, \quad (30)$$

$$\omega = p_\rho - \frac{6}{\lambda}p_\Phi + \frac{6i}{\tilde{\lambda}}p_{\tilde{\Phi}}, \quad (31)$$

the variables separate,

$$H_0 = -\frac{\kappa^2}{12L^6}\omega^2 - \frac{\kappa^2}{L^6}\omega\pi + i\frac{\kappa^2}{L^6}\omega\tilde{\pi} + \frac{1}{2L^6}D\pi^2 + \frac{1}{2L^3}\tilde{D}\tilde{\pi}^2 + Ve^\chi - \tilde{V}e^{i\tilde{\chi}} \quad (32)$$

## Exact classical solutions

The general classical solutions are,

$$\rho = \rho_0 + \kappa^4 \omega t \left( \frac{1}{\tilde{D}} - \frac{1}{D} - \frac{1}{6\kappa^2} \right) + \frac{\kappa^2}{D} \ln \cosh^2(Pt - Q) - \frac{\kappa^2}{\tilde{D}} \ln \cosh^2(\tilde{P}t - \tilde{Q}), \quad (33)$$

$$e^x = \frac{2P^2}{DV} \frac{1}{\cosh^2(Pt - Q)}, \quad e^{i\tilde{x}} \equiv e^\xi = \frac{2\tilde{P}^2}{\tilde{D}\tilde{V}} \frac{1}{\cosh^2(\tilde{P}t - \tilde{Q})}, \quad (34)$$

$$\pi = -\frac{2P}{D} \tanh(Pt - Q) + \frac{\kappa^2}{D} \omega, \quad i\tilde{\pi} = -\frac{2\tilde{P}}{\tilde{D}} \tanh(\tilde{P}t - \tilde{Q}) + \frac{\kappa^2}{\tilde{D}} \omega, \quad (35)$$

$$P^2 = \frac{\kappa^2 \omega^2}{24} (6\kappa^2 + D/2 + CD), \quad \tilde{P}^2 = \frac{\kappa^2 \omega^2}{24} (6\kappa^2 - \tilde{D}/2 + C\tilde{D}), \quad (36)$$

where  $C, Q$  and  $\tilde{Q}$  are arbitrary constants. The solutions are invariant under modified  $\mathcal{PT}$  transformation changing  $\tilde{x}$ .



# The minisuperspace WdW equation

The separation of variables implies,

$$\Psi(\rho, \chi, \tilde{\chi}) = \sum_C \int d\omega e^{iL^3 \omega \rho} \psi(\omega, C, \chi) \tilde{\psi}(\omega, C, \tilde{\chi}), \quad (37)$$

For Hermitian sector we get,

$$\left[ -\frac{\kappa^2}{12} \omega^2 \left( \frac{1}{2} + C \right) + i \frac{\kappa^2}{L^3} \omega \partial_\chi - \frac{D}{2L^6} \partial_\chi^2 + V e^{\chi} \right] \psi = 0, \quad (38)$$

We expect that the norm can be constructed using  $L^2$  on  $\chi$ . Therefore one should select oscillating solutions with  $\nu = \frac{L^3 \kappa \omega}{D} \sqrt{\frac{2}{3} (6\kappa^2 + D/2 + CD)}$

$$\psi(\omega, C, \chi) = \exp \left[ iL^3 \frac{\kappa^2 \omega}{D} \chi \right] K_{i\nu} \left( 2L^3 \sqrt{\frac{2V}{D}} e^{\chi/2} \right) \quad (39)$$

## PTom sector in minisuperspace

In PT sector the equation reads,

$$\left[ -\frac{\kappa^2}{12}\omega^2\left(\frac{1}{2} - C\right) + \frac{\kappa^2}{L^3}\omega\partial_{\tilde{\chi}} - \frac{\tilde{D}}{2L^6}\partial_{\tilde{\chi}}^2 - \tilde{V}e^{i\tilde{\chi}} \right] \tilde{\psi} = 0, \quad (40)$$

with solutions,  $\tilde{\nu} = \frac{L^3\kappa\omega}{\tilde{D}}\sqrt{\frac{2}{3}(6\kappa^2 - \tilde{D}/2 + C\tilde{D})}$

$$\tilde{\psi}(\omega, C, \chi) = \exp\left[L^{3/2}\frac{\kappa^2\omega}{\tilde{D}}\chi\right] J_{\tilde{\nu}}\left(2L^3\sqrt{\frac{2\tilde{V}}{\tilde{D}}}e^{i\tilde{\chi}/2}\right) \quad (41)$$

Two natural options are to require regular behaviour on  $e^{i\tilde{\chi}} = 1$  circle after or before applying exponential factor. In the first option we get to the system considered in (Curtright, Mezincescu, 2007) with biorthogonal system constructed from Neumann polynomials, requiring  $\tilde{\nu}$  to be integer. Then  $\omega$  is real however this way we lose regular behaviour for the function  $\Psi$ . If we require regularity of the whole  $\psi$  we get instead,  $\tilde{\nu} = n + 2iL^{3/2}\frac{\kappa^2\omega}{\tilde{D}}$ . This however would require complex  $\omega$  except when  $n = 0$  which fixes  $C = 1/2$ .

# The fluctuations in the $\kappa \rightarrow 0$ limit

The fluctuation Hamiltonian is greatly simplified when  $\kappa \rightarrow 0$  and quintessence and PTom fluctuations are separated, in  $\kappa \rightarrow 0$  limit it greatly simplifies,

$$H_{2,\Omega} = \Omega^2(\phi^2 + \tilde{\phi}^2) + \frac{D}{2} p_\phi p_\phi + \frac{\tilde{D}}{2} p_{\tilde{\phi}} p_{\tilde{\phi}} + \frac{V}{2} e^\chi \phi^2 - \frac{\tilde{V}}{2} e^{i\tilde{\chi}} \phi^2 + O(\kappa) \quad (42)$$

The Hamiltonian is Hermitian on the purely imaginary classical trajectory  $\tilde{\chi} = i\xi_{class}$ . Let's take the wavepacket concentrated (in terms of  $\mathcal{A}^{PT}\mathcal{A}$ ) near this solution. In  $\kappa \rightarrow 0$  we take as  $\Sigma$  real lines for  $\chi$  and  $\tilde{\chi} - i\xi_{class}$ .

For  $\text{Im}\tilde{\Phi} \neq 0$  the similarity operator  $\eta$  should be introduced but due to the variable separation it should act nontrivially only on  $\tilde{\phi}$  and  $p_{\tilde{\phi}}$  i.e.

$$\eta = \exp \left[ \alpha(t) p_{\tilde{\phi}}^2 + \beta(t) \tilde{\phi}^2 + \gamma(t) (p_{\tilde{\phi}} \tilde{\phi} + \tilde{\phi} p_{\tilde{\phi}}) \right] \quad (43)$$

Then one has to deal with non-Hermitian time-dependent Swanson Hamiltonian - the problem considered in (Fring, Moussa, 2016) and (Maamache et al, 2017)

# The similarity operator

However for trajectories close to  $i\chi_{class}$  one can obtain this operator easily using perturbation theory. We note that the operators in the exponent form the finite dimensional algebra. Then we use,

$$e^{-X_n} \hat{h}_{\vec{\phi}}^{(n)} e^{X_n} = e^{-\text{ad}_{X_n}} \hat{h}_{\vec{\phi}}^{(n)}, \quad e^{-X_n} \partial_{\tau} \left( e^{X_n} \right) = \frac{1 - e^{-\text{ad}_{X_n}}}{\text{ad}_{X_n}} \partial_{\tau} X_n, \quad (44)$$

where the adjoint operator  $\text{ad}_X Y \equiv [X, Y]$  may be represented as a finite-dimensional matrix acting on the algebra.

This gives equations,

$$\partial_{\tau} \alpha_n = 2\gamma_n \tilde{D}, \quad \partial_{\tau} \gamma_n = 2\beta \tilde{D} - 2\alpha (\tilde{V} e^{6\rho - \xi_{class}} + \Omega^2), \quad (45)$$

$$\partial_{\tau} \beta_n = \frac{\tilde{V}}{2} e^{6\rho - \xi_{class}} \delta \tilde{\Phi} - 4\gamma_n (\tilde{V} e^{6\rho - \xi_{class}} + \Omega^2) \quad (46)$$

Which can be solved e.g. numerically near the exact solution  $i\xi_{class}$

# Conclusions

- The pseudo-Hermitian quantum mechanics is a way to deal with apparent instabilities of the phantom models
- The conserved inner product can be constructed in the WKB approximation of the minisuperspace part
- As example we can do this for the model with exponential potentials for sufficiently small  $\Sigma$  near the interesting solution
- This result can be used to construct the probability distributions and check whether the observables for this model show any pathologies

Thank you for your attention!