Parity anomaly in four dimensions

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* Summary.
Parity anomaly in 3D


QED of massless fermions in 3D.

- At the classical level the massless fermionic field $\psi(x)$ satisfies:

$$\left(\not{D}[A]\psi\right)(x) = 0, \quad \not{D}[A] = i\gamma^i \left(\frac{\partial}{\partial x^i} + iA_i(x)\right)$$

- This e.o.m. is invariant upon the parity transformation:

$$\psi(x) \rightarrow \psi(-x), \quad A_i \rightarrow -A_i(-x) \quad \Rightarrow \not{D}[A]\psi(x) \rightarrow -\left(\not{D}[A]\psi\right)(-x) = 0$$
Parity anomaly in 3D

- **Quantization:**

\[
\int D\bar{\psi}D\psi e^{-\int d^3x \bar{\psi}\phi[A]\psi} = \det(\phi[A]) = e^{-W[A]} 
\]

\[
W[A] = -\log \det \phi \quad \text{the one loop effective action}
\]

- The one-loop radiative correction \( W[A] \) breaks the parity symmetry \( A_i(x) \rightarrow -A_i(-x) \) due to the Chern-Simons term!

\[
W[A] = \pm \frac{i}{8\pi} \int d^3x \, A_i \partial_j A_k \epsilon^{ijk} + \text{parity-even terms}
\]


Parity anomaly in 3D

- Let us track the origin of this anomaly.

\[
e^{-W[A]} = \det (\mathcal{D}[A]) = \prod_{\lambda \in \text{Spec} (\mathcal{D}[A])} \lambda
\]

- The parity transformation reflects the spectrum of the Dirac operator:

\[
(\mathcal{D}[A] \psi)(x) = \lambda \psi(x) \rightarrow (-\mathcal{D}[A] \psi)(-x) = \lambda \psi(-x)
\]

- Parity anomaly expresses a spectral asymmetry of the Dirac operator:

\[
W_{\text{odd}} = -\frac{1}{2} \left( \log \det (\mathcal{D}[A]) - \log \det (-\mathcal{D}[A]) \right) \neq 0
\]
Confined fermions in 4D: general setup

We are dealing with the Euclidean 4D manifold $\mathcal{M}$ with a boundary $\partial \mathcal{M}$.

The Dirac operator is the usual one

$$\slashed{D} = i \gamma^\mu (\nabla_\mu + i A_\mu)$$

$$\nabla_\mu = \partial_\mu + \omega_\mu$$

Boundary conditions must satisfy:

- $\slashed{D}^\dagger = \slashed{D}$
- $$(\bar{\psi} \gamma^\mu \psi) n_\mu |_{\partial \mathcal{M}} = 0$$
Confined fermions in 4D: the boundary conditions

We exploit the (Euclidean version of) the MIT bag boundary conditions.

For each component of the boundary $\partial \mathcal{M}_\alpha$ we define the projectors

$$\Pi_{\pm} = \frac{1}{2} \left( 1 \pm i \epsilon_\alpha \gamma^5 \gamma^n \right), \quad \epsilon_\alpha \in \{-1, +1\}$$

and impose the boundary conditions

$$\Pi_{-} \psi |_{\partial \mathcal{M}} = 0$$

These boundary conditions guarantee that:

- $\mathcal{D}^\dagger = \mathcal{D}$
- $(\bar{\psi} \gamma^\mu \psi) n_\mu |_{\partial \mathcal{M}} = 0$


Parity anomaly in 4D: the definition

The parity anomaly can be defined in arbitrary dimension via

\[ W_{\text{odd}} = -\frac{1}{2} \left( \log \det (\mathcal{D}[A]) - \log \det ( -\mathcal{D}[A]) \right) \]

however this expression does not make any sense, unless one introduces the regularisation.

ζ-function regularisation of the fermionic determinant:

\[ W [\mathcal{D}] \equiv - \log \det (\mathcal{D}[A]) \rightarrow \mu^s \Gamma(s) \zeta(s, \mathcal{D}) \equiv W_s [\mathcal{D}], \quad s \to 0 \]

where the zeta function is defined in the following way:

\[ \zeta(s, \mathcal{D}) = \sum_{\lambda > 0} \lambda^{-s} + e^{-i\pi s} \sum_{\lambda < 0} (-\lambda)^{-s} \]
Parity anomaly in 4D: the definition

We are interested in the parity-odd contribution

\[ \zeta_{\text{odd}}(s, D) \equiv \frac{1}{2} \left( \zeta_{\text{odd}}(s, D) - \zeta_{\text{odd}}(s, -D) \right) = \frac{1}{2} \left( 1 - e^{-i\pi s} \right) \eta(s, D), \]

where

\[ \eta(s, D) = \sum_{\lambda > 0} \lambda^{-s} - \sum_{\lambda < 0} (-\lambda)^{-s}. \]

therefore (L. Alvarez-Gaume, S. Della Pietra and G. Moore, 1984)

\[ W_{\text{odd}} = \lim_{s \to 0} \mu^s \Gamma(s) \zeta_{\text{odd}}(s, D) = \frac{i\pi}{2} \eta(0, D) = \text{finite!} \]

Why do we expect it to be different from zero???

- When there is no boundary the spectrum is symmetric with respect to 0, since \( \{D, \gamma^5\} = 0 \).

- Boundary conditions break this property: \( \Pi_- \gamma^5 = \gamma^5 \Pi_+ \)
Parity anomaly in 4D: the computation.

The spectral function $\eta(s, \mathcal{P})$ exhibits the following integral representation:

$$\eta(s, \mathcal{P}) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \tau^s \text{Tr} \left( \mathcal{P} e^{-\tau^2 \mathcal{P}^2} \right)$$

Let us consider the variation of the gauge field $A_\mu(x) \rightarrow A_\mu(x) + \delta A_\mu(x)$:

$$\delta \eta(s, \mathcal{P}) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty d\tau \tau^s \frac{d}{d\tau} \text{Tr} \left( \delta \mathcal{P} \right) \tau e^{-\tau^2 \mathcal{P}^2}.$$

At the physical limit $s = 0$

$$\delta \eta(0, \mathcal{P}) = -\frac{2}{\sqrt{\pi}} \lim_{t \rightarrow +0} \text{Tr} \left( \delta \mathcal{P} \right) \sqrt{t} e^{-t \mathcal{P}^2}$$
Parity anomaly in 4D: the computation.

For an arbitrary matrix-valued function $Q$ the following asymptotic expansion holds at $t \rightarrow +0$:

$$\text{Tr } Q e^{-t \mathcal{D}^2} \simeq \sum_{k=0}^{\infty} t^{k-4} a_k \left( Q, \mathcal{D}^2 \right)$$

The relevant heat kernel coefficients were studied by V. N. Marachevsky and D. V. Vassilevich in Nucl. Phys. B677, 535 (2004)

In our case $a_0$, $a_1$ and $a_2$ vanish, therefore

$$\delta \eta(0, \mathcal{D}) = -\frac{2}{\sqrt{\pi}} a_3 \left( \delta \mathcal{D}, \mathcal{D}^2 \right)$$

$$= \left( -\frac{1}{4\pi^2} \right) \sum_{\alpha} \int_{\partial \mathcal{M}_\alpha} d^3 x \sqrt{\hbar} \epsilon_\alpha \varepsilon^{nabc} (\delta A_a) \partial_b A_c$$
Parity anomaly in 4D: the computation.

The variation of the parity-odd contribution to the one-loop effective action reads:

$$\delta W_{\text{odd}} = \left( -\frac{i}{8\pi} \right) \sum_{\alpha} \int_{\partial M_{\alpha}} d^3x \sqrt{h} \epsilon_\alpha \epsilon^{nabc} (\delta A_a) \partial_b A_c.$$  

In terms of the induced current the answer reads:

$$J^a_{\text{odd}}(x) \equiv \frac{1}{\sqrt{h}} \frac{\delta W_{\text{odd}}}{\delta A_a(x)} = -\frac{i}{8\pi} \epsilon_\alpha(x) \epsilon^{nabc} \partial_b A_c = -\frac{i}{16\pi} \epsilon_\alpha(x) \epsilon^{nabc} F_{bc}$$

If the gauge potential $A$ is defined globally on $M$, we can recover $W_{\text{odd}}$

$$W_{\text{odd}} = \frac{1}{4} \sum_{\alpha} \epsilon_\alpha \left( \frac{i}{4\pi} \int_{\partial M_{\alpha}} \sqrt{h} \epsilon^{nabc} A_a \partial_b A_c \right)$$
Comment on the gauge invariance of the answer.

Let us rewrite the answer in terms of the differential forms. We assume that

\[ A = A_\mu \, dx^\mu \in D^1 (\mathcal{M} \cup \partial \mathcal{M}). \]

Then the answer reads:

\[ W_{\text{odd}} = i \sum_\alpha \text{CS}_\alpha [k_\alpha, A], \]

\[ \text{CS}_\alpha [k_\alpha, A] = \frac{k_\alpha}{4\pi} \int_{\partial \mathcal{M}_\alpha} A \wedge dA, \quad k_\alpha = \varepsilon_\alpha \cdot \frac{1}{4}. \]

Each contribution is invariant upon the gauge transformation:

\[ A \longrightarrow A - iU^{-1}dU, \quad \forall U \in C^\infty (\mathcal{M} \cup \partial \mathcal{M}), \quad |U| = 1. \]

Indeed

\[ d(U^{-1}dU) = 0 \quad \Rightarrow \quad dA \longrightarrow dA, \]

\[ A \wedge dA \longrightarrow A \wedge dA - iU^{-1}dU \wedge dA = A \wedge dA - id (U^{-1}dU \wedge A) \]

\[ \int_{\partial \mathcal{M}_a} A \wedge dA \rightarrow \int_{\partial \mathcal{M}_a} A \wedge dA - i \underbrace{\int_{\partial \mathcal{M}_a} d(U^{-1}dU \wedge A)}_{0} = \int_{\partial \mathcal{M}_a} A \wedge dA. \]
Comment on the classical symmetry.

Let us consider the transformations which leave invariant the classical e.o.m.

\[
\begin{align*}
\{ & i \gamma^\mu \left( \frac{\partial}{\partial x^\mu} + i A_\mu(x) \right) \psi = 0, \\
& \frac{1}{2} \left( 1 - i \epsilon_\alpha \gamma^5 \gamma^n \right) \psi \bigg|_{\partial \mathcal{M}} = 0, \quad \epsilon_\alpha \in \{+1, -1\}.
\end{align*}
\]

The transformation

\[
\begin{align*}
\{ & \psi(x) \rightarrow \psi(-x) \\
& A_\mu(x) \rightarrow -A_\mu(-x)
\end{align*}
\]

is a bad candidate: if \( x \in \mathcal{M} \) there is no guarantee that \(-x \in \mathcal{M}\).

The correct classical symmetry in a presence of the boundary is

\[
\begin{align*}
\{ & \psi(x) \rightarrow \gamma^5 \psi(x) \\
& \epsilon_\alpha \rightarrow -\epsilon_\alpha
\end{align*}
\]

This transformation inverts the nonzero spectrum of the Dirac operator.
Comparison: 3D v.s. 4D results

4D case: \(|k| = \frac{1}{4}\)

3D case: \(|k| = \frac{1}{2}\)

Why is it natural?

Let us consider \(\mathcal{M} = \mathbb{R}^3 \times [0; l]\) at \(l \to 0\).

at \(\epsilon_{\text{up}} = -\epsilon_{\text{down}}\) \(\text{CS}[1/4, A] - \text{CS}[-1/4, A] = \text{CS}[1/2, A] \Rightarrow \text{3D-result}\),

at \(\epsilon_{\text{up}} = +\epsilon_{\text{down}}\) \(\text{CS}[1/4, A] - \text{CS}[+1/4, A] = 0 \Rightarrow \text{nothing}\)

Let us take a look at the spectrums of massless \(I\mathcal{H}\) in both cases at \(A = 0\)

- at \(\epsilon_{\text{up}} = -\epsilon_{\text{down}}\) we obtain \(\lambda^2(p, k_{||}) = k_{||}^2 + m_p^2\), \(m_p^2 = \frac{\pi p^2}{l^2}\), \(p \in \mathbb{Z}\).
  At \(l \to 0\) massless modes with \(p = 0\) survive \(\Rightarrow\) massless 3D spectrum.

- at \(\epsilon_{\text{up}} = +\epsilon_{\text{down}}\) we obtain \(\lambda^2(p, k_{||}) = k_{||}^2 + m_p^2\), \(m_p^2 = \frac{\pi (p+\frac{1}{2})^2}{l^2}\), \(p \in \mathbb{Z}\). At \(l \to 0\) all eigenstates become infinitely massive.
Gravitational contribution: the 3D case.

- What about the gravitational contribution to the parity anomaly?

\[ \mathcal{D} = i \gamma^i \nabla_i, \quad \nabla_i = \partial_i + \omega_i \]

\[ W_{\text{odd}} = - \left( \log \det (\mathcal{D}) - \log \det (-\mathcal{D}) \right) \neq 0? \]

- The answer is “yes”, it is different from zero.

\[ W_{\text{odd}} = -\frac{ik}{4\pi} \int d^3x \sqrt{g} \epsilon^{\mu\nu\rho} \left( \Gamma^\lambda_{\mu\kappa} \partial_{\nu} \Gamma^\kappa_{\rho\lambda} + \frac{2}{3} \Gamma^\lambda_{\mu\kappa} \Gamma^\kappa_{\nu\sigma} \Gamma^\sigma_{\rho\lambda} \right). \]

- There have been contradicting results in the literature regarding the coefficient $k$ in front of Chern-Simons term.

\[ k = \frac{1}{48}, \quad \text{(Goni, Valle -1986; Vuorio -1986; van der Bij - 1986)} \]

\[ k = \frac{1}{16}, \quad \text{(Ojima -1989)} \]
Gravitational contribution to the 4D parity-anomaly.

We are dealing with the Euclidean 4D manifold $\mathcal{M}$ with a boundary $\partial \mathcal{M}$.

The Dirac operator is the usual one with the bag boundary conditions:

$$\begin{aligned}
\left\{ \begin{array}{l}
\mathcal{D} = i \gamma^\mu \left( \partial_\mu + \omega_\mu \right) \\
\frac{1}{2} \left( 1 - i \epsilon_\alpha \gamma^5 \gamma^n \right) \psi \bigg|_{\partial \mathcal{M}} = 0,
\end{array} \right.
\end{aligned}$$

$\epsilon_\alpha \in \{+1, -1\}$.

Upon the zeta-function regularization the parity anomaly reads:

$$W_{\text{odd}} = \frac{i \pi}{2} \eta(0, \mathcal{D}),$$

where

$$\eta(s, \mathcal{D}) = \sum_{\lambda > 0} \lambda^{-s} - \sum_{\lambda < 0} (-\lambda)^{-s}.$$
Parity anomaly in 4D: the computation.

Using again the integral representation:

\[ \eta(s, \dot{\phi}) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \tau^s \text{Tr} \left( \dot{\phi} e^{-\tau^2\dot{\phi}^2} \right) \]

Let us consider the variation of the vierbeins

\[ e_{\mu a} \rightarrow e_{\mu a} + \delta e_{\mu a} : \]

\[ \delta \eta(s, \dot{\phi}) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty d\tau \tau^s \frac{d}{d\tau} \text{Tr} \left( \delta \dot{\phi} \right) e^{-\tau^2\dot{\phi}^2}, \]

\[ \delta \dot{\phi} = i \gamma^\mu \delta \omega_\mu + i (\delta e_\mu^a) \gamma^a \nabla_\mu - 1\text{-st order diff. operator!} \]

At the physical limit \( s = 0 \)

\[ \delta \eta(0, \dot{\phi}) = -\frac{2}{\sqrt{\pi}} \lim_{t \rightarrow +0} \text{Tr} \left( \delta \dot{\phi} \right) \sqrt{t} e^{-t\dot{\phi}^2} \]
Parity anomaly in 4D: the computation.

For the first order diff op. \( Q = Q^\mu_1 \partial_\mu + Q_0 \) the asymptotic expansion at \( t \to +0 \) has a different structure:

\[
\text{Tr } Q e^{-t\Phi^2} \simeq \sum_{k=-1}^\infty t^{k-4/2} a_k (Q, \Phi^2)
\]

There is a trick which allows to compute \( a_k(Q, \Phi^2) \) using the known expressions for \( a_{k+2}(\bar{Q}, \mathcal{L}) \), where \( \mathcal{L} \) is a generic Laplace-type operator and \( \bar{Q} \) is a matrix valued function, see JHEP 1803 (2018) 072 by M.K. and D.Vassilevich.

In our case \( a_{-1}, a_0, a_1 \) and \( a_2 \) vanish, therefore

\[
\delta W_{\text{odd}} = -i \sqrt{\pi} a_3 (\delta \Phi, \Phi^2) = \int_{\partial M} d^3 x \sqrt{h} \varepsilon_\alpha \left\{- \frac{i}{384\pi} (\delta g_{jq}) \bar{R}_{sp} \frac{q^k}{k} \varepsilon^{njsp} \right. \\
+ \left. \frac{i}{256\pi} \left( (\delta g_{si})_{;n} K^i_{p;l} - (\delta g_{si}) (K^i_{l} K^r_{p;r} + K^r_{p} K^i_{l;r} + K^r_{si} K_{rp;l}) \right) \right\} \varepsilon^{nspl}.
\]
Gravitational contribution: the 4D the answer.

The solution of the variational equation reads:

\[ W_{\text{odd}} = -\frac{i}{4\pi} \frac{1}{96} \int_{\partial \mathcal{M}} d^3 x \sqrt{h} \varepsilon_\alpha \left[ \left( \tilde{\Gamma}_{r}^{q} \partial_{j} \tilde{\Gamma}_{r}^{q} + \frac{2}{3} \tilde{\Gamma}_{r}^{q} \tilde{\Gamma}_{p}^{q} \tilde{\Gamma}_{r}^{p} \right) \epsilon^{nijk} + \frac{3}{2} K_{s} K_{l}^{i} \epsilon^{nspl} \right] \]

\[ P = \frac{1}{4} \int_{\mathcal{M}} d^4 x \sqrt{g} \epsilon^{\mu \nu \alpha \beta} R^{\sigma} \tau \mu \tau \nu \sigma \alpha \beta \]

\[ = -\int_{\partial \mathcal{M}} d^3 x \sqrt{h} \left[ \left( \tilde{\Gamma}_{i}^{m} \partial_{j} \tilde{\Gamma}_{k}^{l} + \frac{2}{3} \tilde{\Gamma}_{i}^{m} \tilde{\Gamma}_{j}^{p} \tilde{\Gamma}_{k}^{l} \right) \epsilon^{nijk} - 2K_{i} K_{k}^{l} \epsilon^{nijk} \right] \]

• This answer is invariant upon the local Weyl transformations:

\[ g_{\mu \nu} \longrightarrow e^{2\phi} g_{\mu \nu} \]

• The coefficient \( \frac{1}{96} \) in from of the Chern-Simons term is exactly twice smaller than the corresponding coefficient in the 3D case.

• It has no relation to the (bulk) Pontryagin type topological density, regardless of the choice of the sign factors \( \varepsilon_\alpha \).
Summary.

- We considered the massless QED.

- If one traps fermions inside the $4D$ manifold with a boundary, the one loop radiative corrections induce the Chern-Simons term on the boundary.

- This Chern-Simons term comes out from the spectral asymmetry of the Dirac operator due to the boundary conditions. Presence of such an asymmetry represents the parity anomaly.

- The level of this induced Chern-Simons term is exactly twice smaller than in the 3D case.

- Apart from that the P-odd radiative corrections induce the gravitational Chern-Simons term. The overall coefficient is again twice smaller than in the 3D case. The main novelty in the 4D setup is a presence of the very specific contribution, which depends on the extrinsic curvature.