

Ultraviolet phenomena in AdS self-interacting quantum field theory

ETA, U.Moschella and F.Popov

[arXiv:1802.02955](https://arxiv.org/abs/1802.02955)

Usually in QFT one assumes that

- UV phenomena are local
- UV renormalization can be done via analytical continuation from Euclidian to Minkowskian signature
- However QFT in curved space–time is full of surprises:
dS — in IR, AdS — in UV.

- We consider scalar field theory:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right].$$

- The one loop contribution to the effective action of the theory:

$$\Gamma^{(4)} = -i \frac{3\lambda^2}{2(4\pi^2)^2} \int d^4x \int d^4y \phi^2(x) \phi^2(y) F^2(x-y).$$

- Here

$$F(x) \approx \frac{1}{4\pi^2} \frac{i}{x^2 - i\epsilon}$$

is **the most singular part** of the Feynman propagator in position space when $x^2 \rightarrow 0$.

UV renormalization in x -space

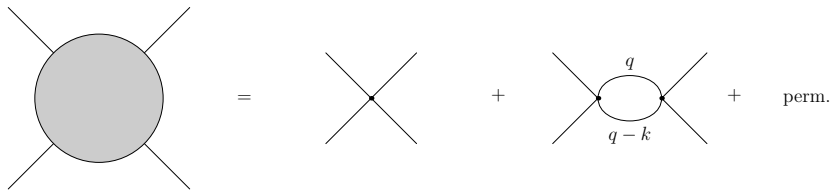


Figure: The one-loop corrections to the four-point correlation function in the interacting scalar field theory.

UV renormalization in x -space

- We may extract the leading divergent contribution by changing the variables $x^\mu = X^\mu + \frac{z^\mu}{2}$, $y^\mu = X^\mu - \frac{z^\mu}{2}$, $\mu = 0, 1, 2, 3$ and by diagonally expanding $\phi^2(X + z/2) \phi^2(X - z/2)$
- In fact,

$$\Gamma^{(4)} = -i \frac{3\lambda^2}{2(4\pi^2)^2} \int d^4X \phi^4(X) \int d^4z \frac{1}{(z^2 - i\epsilon)^2} + \text{finite terms}$$

The z -integral provides the standard logarithmic UV divergence.

- Note the importance of the proper $i\epsilon$ prescription. If one replaces the **Feynman** propagator with the **Wightman** function, then:

$$\int d^4z \frac{1}{[(z_0 - i\epsilon)^2 - \vec{z}^2]^2} = 0.$$

A simple example

- We continue by considering the above theory in flat space-time but in the presence of **an ideal mirror** placed at $x_3 = 0$. **The ideal mirror reflects all the modes equally well, irrespectively of their momenta.** This is expressed by the boundary condition $\phi|_{x_3=0} = 0$ at the mirror.
- **A real physical mirror** is definitely transparent to very high energy modes. On general physical grounds one can expect that, if a is a characteristic interatomic distance of the material of the mirror, a mode whose wavelength k is much larger than $1/a$ will not see the mirror at all.
- **A real physical mirror** can be modeled by a potential barrier which reflects some of the modes and is transparent to the other ones, e.g.:

$$[\square + m^2] \phi = \alpha \delta(x_3) \phi.$$

A simple example

- The most singular part of the Feynman propagator in the presence of an ideal mirror is the following distribution

$$F_{mir}(x, y) \approx \frac{i}{4\pi^2} \frac{1}{s - i\epsilon} - \frac{i}{4\pi^2} \frac{1}{\bar{s} - i\epsilon},$$

where $s = (x - y)^2$ and $\bar{s} = (x - \bar{y})^2$ and \bar{y} is the mirror image of the source point y .

- In **Euclidean signature**, s vanishes only when $x = y$ and \bar{s} only when $x = \bar{y}$. But the point \bar{y} does not belong to the portion of space-time that we are considering, $x_3 > 0$ and $y_3 > 0$. Hence, inside the loops in Euclidean signature \bar{y} plays no role.
- In **Lorentzian signature**, s and \bar{s} vanish on the light-cones whose tips are y and, respectively, \bar{y} . Therefore, even though \bar{y} does not belong to the space-time manifold its light-cone penetrates into it.

A simple example

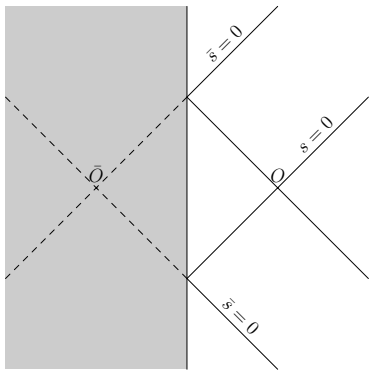


Figure: Light cones in the case of the ideal mirror.

A simple example

- The first singularity in $F_{mir}(x, y)$ provides the same contribution as in empty space. As regards the second term we have to do the following diagonal expansion:
 $x^\mu - \bar{y}^\mu = z^\mu, x^\mu + \bar{y}^\mu = 2X^\mu, \mu = 0, 1, 2, 3.$
- Then, the effective action contains the following term:

$$\Gamma_{\bar{s}=0}^{(4)} = -i \frac{3\lambda^2}{2(4\pi^2)^2} \int d^4X \int_{z_3 \geq |X_3|} d^4z \frac{\phi^2(X + z/2) \phi^2(\bar{X} - \bar{z}/2)}{(z^2 - i\epsilon)^2}.$$

- Even though z^2 may vanish the components of four-vector z are generically not small and the diagonal expansion of $\phi^2(X + z/2) \phi^2(\bar{X} - \bar{z}/2)$ cannot be performed.
- Both the singularities of $F_{mir}(x, y)$ at $s = 0$ and $\bar{s} = 0$ do contribute to the UV divergence of the integral on the right hand side, while the mixed terms contribute finite expressions.

The geometry of AdS and Lobachevsky spaces

- The 4-dimensional **Euclidean AdS (Lobachevsky)** space is one of the sheets (say $X_0 \geq 1$) of the two-sheeted real hyperboloid

$$EAdS_4 := \{X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = 1\},$$

embedded into:

$$ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2.$$

- The 4-dimensional **(Lorentzian) AdS** space is the hyperboloid:

$$AdS_4 := \{X_0^2 - X_1^2 - X_2^2 - X_3^2 + X_4^2 = 1\}$$

embedded into

$$ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 + dX_4^2.$$

- The EAdS and AdS are related to each other via the analytic continuation $X_4 \rightarrow i X_4$. We set the curvatures of the hyperboloids to one: $R = 1$.

The geometry of the 2D Lorentzian global AdS space

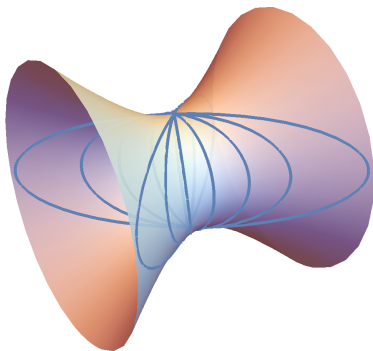


Figure: The timelike geodesics issued from a point Y focus at the antipodal point $-Y$. The boundary acts somehow like a (thick) mirror for massive particles. **Time is compact.**

The geometry of AdS and Lobachevsky spaces

- The hyperbolic distance is defined via the **invariant scalar product**:

$$\xi = \eta_{\mu\nu} X^\mu Y^\nu = \cosh d(X, Y),$$

where $d(X, Y)$ is the geodesic distance.

- In **EAdS** the hyperbolic distance is $\xi \geq 1$, because $d(X, Y)$ is real, while in **AdS** ξ can take any real value.
- The Feynman propagator obeys

$$\begin{aligned} [\square + m^2] F(X, Y) &= [(1 - \xi^2) \partial_\xi^2 - 4 \xi \partial_\xi + m^2] F(\xi) = \\ &= 4 \pi \delta(X, Y) + 4 \pi i e^{-i\pi\nu} \delta(X, -Y). \end{aligned}$$

The Feynman propagator in AdS and Lobachevsky spaces

- The Feynman propagator can be represented both in **EAdS**, in **global AdS** and in its covering $\widetilde{\text{AdS}}$ manifold as follows:

$$F(X, Y) = A_+ {}_1F_2 \left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; \frac{1 + \xi + i\epsilon}{2} \right) + A_- {}_1F_2 \left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; \frac{1 - \xi - i\epsilon}{2} \right)$$

- $\nu = \sqrt{\frac{9}{4} + m^2}$
- When $\xi^2 \rightarrow 1$ there is the following leading singularities of the AdS Feynman propagator:

$$F(\xi) \approx -\frac{i}{8\pi^2 (\xi - 1 + i\epsilon)} - \frac{e^{-i\pi\nu}}{8\pi^2 (\xi + 1 + i\epsilon)}.$$

The Feynman propagator in AdS and Lobachevsky spaces

- The first singularity, at $\xi = 1$, is the same as in flat empty space. Note that:

$$(X - Y)^2 - i\epsilon = 2(1 - (\xi + i\epsilon))$$

- The second singularity, at $\xi = -1$, is when X sits on the light cone with the apex at $\tilde{Y} = -Y$ — point antipodal to Y .
- In **Lobachevsky space** the second singularity is not seen, because there $\xi \geq 1$. But in **AdS** the second singularity is present.

Other options of $i\epsilon$ -prescriptions

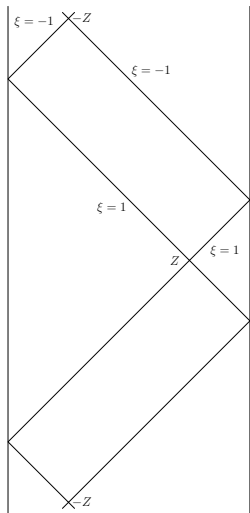


Figure: Covering space of the global AdS

Loop corrections in AdS and Lobachevsky spaces

- In global AdS the relevant part of the correction is:

$$\Gamma^{(4)} \propto \lambda^2 \int d^5 X \delta(X^2 - 1) \int d^5 Y \delta(Y^2 - 1) \phi^2(X) \phi^2(Y) \times \\ \times \left[\frac{1}{(X - Y)^2 - i\epsilon} - \frac{i e^{-i\pi\nu}}{(X + Y)^2 - i\epsilon} \right]^2.$$

- The first pole leads to the same renormalization as in flat spacetime. The second pole is different and leads to divergences of a new type. The cross terms lead to less singular contributions.
- Thus, we have to introduce a new counter-term into the Lagrangian:

$$\Delta\mathcal{L} = \frac{\gamma e^{-2\pi i\nu}}{4} \phi^2(X) \phi^2(-X),$$

with a complex coefficient depending on the mass parameter and a new coupling constant γ .

Conclusions

- The main result of this paper is that the UV renormalization of the four–point function in AdS space–time generates non–local counter terms that respect the isometry group.
- There seems to be no local measurement which allows to detect terms such as the one found above and their presence does not destroy the renormalizability of the theory (although they do affect the beta–function).
- Obviously one can discard the situation of the perfect mirror as unphysical, but still accept the effects discussed in this paper for AdS space as physical (despite the presence of the complex coupling constant).
- Another possibility would be to define quantum field theory in AdS space via the analytical continuation from Lobachevsky space, i.e. from the Euclidean manifold. However, such an analytical continuation does not allow to address the issues of e.g. non–stationary phenomena within the AdS/CFT correspondence (at least beyond the $1/N$ approximation).