Ghost-Free Theory with Third-Order Time Derivatives

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$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

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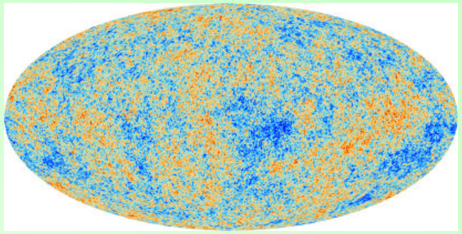
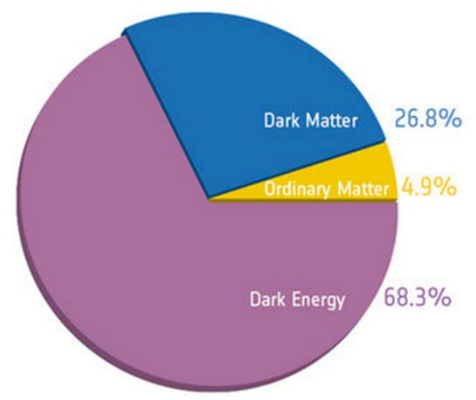
Introduction

Inflation & dark energy

Inflation & dark energy (cosmic acceleration in early and current Universe) are strongly supported by observations.





PLANCK

In order to identify the inflaton and the source of dark energy, it is quite useful to consider the most general models based on a scalar tensor theory.

Then, we can constrain models (or to single out the true model finally) from the observational results.

The following question arises:

What is the most general
scalar-tensor theory without ghost ?

How widely can we extend scalar tensor theory ?

- **A kinetic term of an inflaton is not necessarily canonical.**

$$\mathcal{L} = X - V(\phi), \quad X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \Longrightarrow \mathcal{L} = K(\phi, X)$$

(k-inflation)

(Armendariz-Picon et.al. 1999)

- **An inflaton is not necessarily minimally coupled to gravity.**

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}M_G^2 R + \mathcal{L}_\phi\right) \Longrightarrow \Delta S = \int d^4x\sqrt{-g}f(\phi)R$$

(Higgs inflation)

(Cervantes-Cota & Dehnen 1995, Bezrukov & M. Shaposhnikov 2008)

- **Action may include higher derivatives.** (Nicolis et.al. 2009)

$$\mathcal{L} = K(\phi, X) \Longrightarrow \Delta\mathcal{L} = G(\phi, X)\Box\phi$$

Theories with higher order derivatives are quite dangerous in general.

Example with higher order (time) derivatives

- $L = \frac{1}{2}\ddot{q}^2(t) \quad \xrightarrow{\text{EL eq.}} \quad q^{(4)} = 0$ **requires 4 initial conditions.**

  $\Updownarrow$

  2 (real) DOF

- $L_{\mathrm{eq}}^{(1)} = \ddot{q}u - \frac{1}{2}u^2 \quad \xrightarrow{\text{EL eq.}} \quad \left\{ \begin{array}{rcl} \ddot{u} & = & 0, \\ \ddot{q} & = & u, \end{array} \right. \quad \Longrightarrow \quad q^{(4)} = 0$

$$x \equiv \frac{q-u}{\sqrt{2}}, \quad y \equiv \frac{q+u}{\sqrt{2}} \quad \Longrightarrow \quad L_{\mathrm{eq}}^{(1)} = -\dot{q}\dot{u} - \frac{1}{2}u^2 = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2 - \frac{1}{4}(x-y)^2.$$

$$\Longrightarrow \quad H = \frac{1}{2}p_x^2 - \frac{1}{2}p_y^2 + \frac{1}{4}(x-y)^2.$$

$$(p_x \equiv \dot{x}, \quad p_y \equiv \dot{y})$$

2 (real) DOF = 1 healthy & 1 ghost

- $L_{\mathrm{eq}}^{(2)} = \frac{1}{2}\dot{Q}^2 + \lambda(Q - \dot{q}) \quad \Longrightarrow \quad p \equiv \frac{\partial L_{\mathrm{eq}}^{(2)}}{\partial \dot{q}} = -\lambda, \quad P \equiv \frac{\partial L_{\mathrm{eq}}^{(2)}}{\partial \dot{Q}} = \dot{Q}.$

$$\Longrightarrow \quad H = p\dot{q} + P\dot{Q} - L_{\mathrm{eq}}^{(2)} = \frac{1}{2}P^2 + pQ.$$

Hamiltonian is unbounded through a linear momentum !!

Ostrogradski's theorem

(Ostrogradsky 1850)

Assume that $L = L(\ddot{q}, \dot{q}, q)$ **and** $\frac{\partial L}{\partial \ddot{q}}$ **depends on** $\ddot{q}$ **:**

(Non-degeneracy)

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}^2 t}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0, \quad \Longrightarrow \quad q^{(4)} = q^{(4)}\left(q^{(3)}, \ddot{q}, \dot{q}, q\right).$$

Canonical variables :

$$\begin{cases} q, & p := \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} \quad \left(= \frac{\partial L_{\mathrm{eq}}}{\partial \dot{q}}\right), \\ Q := \dot{q}, & P := \frac{\partial L}{\partial \ddot{q}} \quad \left(= \frac{\partial L_{\mathrm{eq}}}{\partial \dot{Q}}\right). \end{cases}$$

$$L_{\mathrm{eq}} = L(\dot{Q}, \dot{q}, q) + \lambda(Q - \dot{q})$$

Non-degeneracy $\Leftrightarrow$ $\ddot{q} = \ddot{q}\left(q, \dot{q}, \frac{\partial L}{\partial \ddot{q}}\right)$ $\Leftrightarrow$ $\dot{Q} = \ddot{q} = \ddot{q}\,(q, Q, P)$

Hamiltonian:

$$\begin{aligned} H(q, Q, p, P) &:= p\dot{q} + P\dot{Q} - L \\ &= pQ + P\ddot{q}(q, Q, P) - L(q, Q, \ddot{q}(q, Q, P)). \end{aligned}$$

p depends linearly on H so that no system of this form can be stable !!

N.B. $\frac{\partial L}{\partial \phi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \phi)}\right) + \partial_\mu \partial_\nu\left(\frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi)}\right) = 0.$ $\Longrightarrow$ $\frac{i}{(p^2 + m_1^2)(p^2 + m_2^2)} = \frac{1}{m_2^2 - m_1^2}\left(\frac{i}{p^2 + m_1^2} - \frac{i}{p^2 + m_2^2}\right).$

(propagators)

How to circumvent Ostrogradsky's arguments to obtain healthy higher order derivative theories ?

Loophole of Ostrogradski's theorem

We can break the non-degeneracy condition which requires that $\frac{\partial L}{\partial \ddot{q}}$ depends on ddot{q}.

(NB: another interesting possibility is infinite derivative theory)

In case Lagrangian depends on only a position q and its velocity dot{q}, degeneracy implies that EOM is first order, which represents not the dynamics but the constraint.

$$\Updownarrow$$

In case Lagrangian depends on q, dot{q}, ddot{q}, degeneracy implies that EOM can be (more than) second order, which can represent the dynamics.

Generalized Galileon = Horndeski

Deffayet et al. 2009, 2011 — Horndeski 1974

equivalence

Kobayashi, MY, Yokoyama 2011

$$
\left\{
\begin{aligned}
\mathcal{L}_2 &= \boxed{K(\phi, X)}, \\
\mathcal{L}_3 &= -\boxed{G_3(\phi, X)}\Box\phi, \\
\mathcal{L}_4 &= \boxed{G_4(\phi, X)}R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\
\mathcal{L}_5 &= \boxed{G_5(\phi, X)}G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\
&\quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].
\end{aligned}
\right.
$$

$$
X = -\frac{1}{2}(\nabla\phi)^2, \qquad G_{iX} \equiv \partial G_i/\partial X.
$$

This is the most general scalar tensor theory whose Euler-Lagrange EOMs are up to second order though the action includes second derivatives.
Many of inflation and dark energy models can be understood in a unified manner.

NB :

- **$G_4 = M_G^2 / 2$ yields the Einstein-Hilbert action**
- **$G_4 = f(\varphi)$ yields a non-minimal coupling of the form $f(\varphi)R$**
- **The new Higgs inflation with $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ comes from $G_5 \propto \varphi$ after integration by parts.**

Beyond Horndeski theory

Gleyzes, Langlois, Piazza, and Vernizzi (GLPV) pointed out that there is extended theory with the number of propagating degrees of freedom unchanged, even though apparent EOMs of the theory are higher (third) order.

(Gleyzes et al. 2014, Zumalacarregui & Garcia-Bellido 2014)

Noui and Langlois pointed out the importance of degeneracy of kinetic matrix of terms with different order derivatives and proposed degenerate higher order scalar-tensor (DHOST) theory.

(Noui & Langlois 2016, etc)

Another direction is to consider infinitely many higher derivatives (nonlocal theory). (Barnaby & Kamran 2008, References therein)

e.g. $e^{-\Box/\Lambda^2}(\Box - m^2)\phi = 0. \Longrightarrow \dfrac{i}{e^{p^2/\Lambda^2}(p^2+m^2)}$ Only a pole of propagator appears with $p^2 = m^2$.

(Biswas, A. Mazumdar, W. Siegel 2006)

We are interested in an issue whether further extension is possible.

As far as I know, all of the (finite) higher order derivative theories without ghosts include up to second time derivatives.

(Xian Gao proposed arbitrary higher spatial derivative theory in 2014.)

What happens if we consider "third" order time derivative theories ?

(Is it straightforward extension of second order time derivative theories ???)

Healthy degenerate theories with second order derivatives for point particles

(as a first step with keeping in mind future extension to scalar-tensor theory)

Lagrangian up to second order derivatives

(Motohashi, Noui, Suyama, MY, Langlois. 2016)

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i) \;\; (a = 1, \cdots, n; \;\; i = 1, \cdots, m)$$

Variables with second order derivatives: (φ^a)
Variables with first order derivatives: (q^i)

φ^a and q^i generically obey fourth order and second order equations of motion, respectively ➔ Ostrogradsky instability

Let us derive the conditions to escape such an instability by Hamiltonian analysis, starting from the equivalent Lagrangian:

$$L^{(1)}_{eq}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\underset{\substack{\uparrow \\ \ddot{\phi}^a}}{\dot{Q}^a}, \underset{\substack{\uparrow \\ \dot{\phi}^a}}{Q^a}, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

Hamiltonian analysis

$$L^{(1)}_{eq}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

$\dot{\phi}^a$

(3n+m) canonical variables : Q^a, φ^a, q^i, ~~χ^a~~

(3n+m) canonical momenta : Pa, πa, pi, ~~ρa~~

(4n+2m) phase DOF

n ghosts!!

$$P_a = \frac{\partial L}{\partial \dot{Q}^a} \equiv L_{\dot{Q}} \ , \quad \pi_a = \frac{\partial L^{(1)}_{eq}}{\partial \dot{\phi}^a} = \lambda_a \ , \quad p_i = \frac{\partial L}{\partial \dot{q}^i} \equiv L_{\dot{q}} \ , \quad \rho_a = \frac{\partial L^{(1)}_{eq}}{\partial \dot{\lambda}^a} = 0 \, .$$

Two sets of n primary constraints:

$$\Phi_a = \pi_a - \lambda_a \approx 0 \ , \quad \Psi_a = \rho_a \approx 0 \, ,$$

$$H = H_0 + \pi_a Q^a \quad \text{with} \quad H_0 = P_a \dot{Q}^a + p_i \dot{q}^i - L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i)$$

π^a appears linearly in Hamiltonian and hence it is unbounded if the system is nondegenerate without further primary constraints.

Degenerate Lagrangian

For healthy theories, we have to eliminate n DOF from constraints

$$\begin{pmatrix} \delta P_a \\ \delta p_i \end{pmatrix} = K \begin{pmatrix} \delta \dot{Q}^b \\ \delta \dot{q}^j \end{pmatrix}, \quad K \equiv \begin{pmatrix} L_{\dot{Q}^a \dot{Q}^b} & L_{\dot{Q}^a \dot{q}^j} \\ L_{\dot{q}^i \dot{Q}^b} & L_{\dot{q}^i \dot{q}^j} \end{pmatrix} = \begin{pmatrix} L_{ab} & L_{aj} \\ L_{ib} & L_{ij} \end{pmatrix}$$

Kinetic matrix must be degenerate !!

$$\Longrightarrow \quad L_{ab} - L_{ai} L^{ij} L_{jb} = 0 \qquad \text{(First DC)}$$

Assume det Lij ≠ 0 (q^i is a normal variable)

$$\because \quad \left(\det K = \det(L_{ab} - L_{ai} L^{ij} L_{jb}) \det L_{ij} \right)$$

(all n eigenvalues are zero) (m non-zero eigenvalues)

$$\Longleftrightarrow \quad \Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

Assume det Lij ≠ 0

(Additional primary constraints)

Additional primary constraints

(Motohashi & Suyama 2014)

$$\Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

$\Rightarrow$ $H_T = H(P_a, p_i, \pi_a, Q^a, q^i, \phi^a) + \mu^a \Phi_a + \nu^a \Psi_a + \xi^a \Xi_a$

$(\Phi_a = \pi_a - \lambda_a \approx 0, \ \ \Psi_a = \rho_a \approx 0)$

$$\Rightarrow \left\{ \begin{aligned} &\dot{\Phi}_a = \{\Phi_a, H\} - \nu^a + \xi^b \partial F_b/\partial \phi^a \approx 0, \\ &\dot{\Psi}_a = \mu^a \approx 0, \\ &\dot{\Xi}_a = \{\Xi_a, H\} + \xi^b \underbrace{\{\Xi_a, \Xi_b\}}_{\equiv \, M_{ab}} \approx 0. \end{aligned} \right.$$

$(\{\Phi_a, \Xi_b\} = \partial F_b/\partial \phi^a)$ $\Rightarrow$ **Fix μ^a & ν^a**

If det $M_{ab} \neq 0$, all ξ^a are fixed and no secondary constraints.

$\Rightarrow$ **Not sufficient number of constraints to eliminate all ghosts.**

$\Rightarrow$ **Simplest case for healthy theory : $M_{ab} = 0$ (second DC)**

$\Rightarrow$ $\Theta_a \equiv \{\Xi_a, H\} = -\pi_a + \{\Xi_a, H_0\} - \partial F_a/\partial \phi_b Q^b \approx 0$

(New constraint, which fixes all π_a in terms of the other phase space variables)

Summary of second order system

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i)$$

$$\Longleftrightarrow \quad L_{eq}^{(1)}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

(3n+m) canonical variables : $Q^a \swarrow \dot{\phi}^a$, φ^a, q^i, $\cancel{\lambda}^a$

(3n+m) canonical momenta : $\cancel{P}a$, $\cancel{\pi}a$, pi, $\cancel{\rho}a$

$\Rightarrow$ **(2n+2m) phase DOF** **no ghosts!!**

- **First DC :** $L_{ab} - L_{ai}L^{ij}L_{jb} = 0$

$$\left[\begin{pmatrix}\delta P_a \\ \delta p_i\end{pmatrix} = K \begin{pmatrix}\delta \dot{Q}^b \\ \delta \dot{q}^j\end{pmatrix}, \quad K \equiv \begin{pmatrix} L_{\dot{Q}^a\dot{Q}^b} & L_{\dot{Q}^a\dot{q}^j} \\ L_{\dot{q}^i\dot{Q}^b} & L_{\dot{q}^i\dot{q}^j}\end{pmatrix} = \begin{pmatrix} L_{ab} & L_{aj} \\ L_{ib} & L_{ij}\end{pmatrix}\right]$$

Assume det $L_{ij} \neq 0$

$\Rightarrow$ $\Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$ **(primary constraints)**

- **Second DC :** $M_{ab} = \{\Xi_a, \Xi_b\} = -\frac{\partial F_a}{\partial Q^b} + \frac{\partial F_b}{\partial Q^a} + \frac{\partial F_a}{\partial q_i}\frac{\partial F_b}{\partial p_i} - \frac{\partial F_a}{\partial p_i}\frac{\partial F_b}{\partial q_i} = 0.$

$\Rightarrow$ $\Theta_a \equiv \{\Xi_a, H\} = -\pi_a + \{\Xi_a, H_0\} - \partial F_a/\partial\phi_b\, Q^b \approx 0$ **(secondary CS)**

$H = H_0 + \pi_a Q^a$ with $H_0 = P_a\dot{Q}^a + p_a\dot{q}^a - L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i)$

EOMs : $(L_{ab} - L_{ai}L^{ij}L_{jb})\phi^{b\,(4)} + M_{ab}\phi^{b\,(3)} + \cdots = 0$ $\Rightarrow$ **second order**

Is the extension to third order (time) derivative system straightforward ???

No essential difference ???

➡ **There is big difference !!**

Quadratic model with third order derivatives

(Motohashi, Suyama, MY 2018)

$$L = \frac{a_{nm}}{2}\dddot{\psi}^n\dddot{\psi}^m + \frac{b_{nm}}{2}\ddot{\psi}^n\ddot{\psi}^m + \frac{c_{nm}}{2}\dot{\psi}^n\dot{\psi}^m + \frac{d_{nm}}{2}\psi^n\psi^m + e_{nm}\dddot{\psi}^n\ddot{\psi}^m + f_{nm}\ddot{\psi}^n\dot{\psi}^m + \frac{A_{ij}}{2}\dot{q}^i\dot{q}^j + \frac{B_{ij}}{2}q^iq^j + C_{ij}\dot{q}^iq^j + \alpha_{ni}\dddot{\psi}^n\dot{q}^i\,.$$

$(n = 1, \cdots, N)$

$(i = 1, \cdots, I)$

$$\left(\psi^n = \psi^n(t),\ q^i = q^i(t), \quad a_{nm}, b_{nm}, \cdots : \text{const}, \quad \det A_{ij} \neq 0\right)$$

$$\Longleftrightarrow \quad L_{\text{eq}} = L(\underset{\substack{\uparrow\\ \dddot{\psi}^n}}{\dot{Q}^n}, \underset{\substack{\uparrow\\ \ddot{\psi}^n}}{Q^n}, \underset{\substack{\uparrow\\ \dot{\psi}^n}}{R^n}, \psi^n, \dot{q}^i, q^i) + \xi_n(\dot{\psi}^n - R^n) + \lambda_n(\dot{R}^n - Q^n).$$

Canonical momenta for $(Q^n, R^n, \psi^n, q^i, \lambda_n, \xi_n)$ **:**

$$P_{Q^n} = a_{nm}\dot{Q}^m + \alpha_{ni}\dot{q}^i + e_{nm}Q^m,$$
$$p_i = \alpha_{ni}\dot{Q}^n + A_{ij}\dot{q}^j + C_{ij}q^j,$$

$$\boxed{\begin{aligned} &P_{R^n} = \lambda_n, \quad \pi_{\psi^n} = \xi_n,\\ &\rho_{\lambda_n} = 0, \quad \rho_{\xi_n} = 0. \end{aligned}}$$

Primary constraints

Linear dependence of momenta on Hamiltonian

Canonical variables : $\left(\underset{\ddot{\psi}^n}{Q^n}, \underset{\dot{\psi}^n}{R^n}, \psi^n, q^i, \cancel{\lambda_n}, \cancel{\xi_n}\right)$

Canonical momenta : $\left(P_{Q^n}, P_{R^n}, \pi_{\psi^n}, p^i, \cancel{\rho_{\lambda_n}}, \cancel{\rho_{\xi_n}}\right)$

primary constraints

$$\Rightarrow \quad H = H_0 + P_{R^n} Q^n + \pi_{\psi^n} R^n .$$

$\left(H_0 = \frac{1}{2} A^{ij} \tilde{p}_i \tilde{p}_j - \frac{1}{2} b_{nm} Q^n Q^m - \frac{1}{2} c_{nm} R^n R^m - \frac{1}{2} d_{nm} \psi^n \psi^m - f_{nm} Q^n R^m - \frac{1}{2} B_{ij} q^i q^j \ : \right.$ no dependece on $\left. P_{R^n}, \pi_{\psi^n} \right)$

Due to the linear dependence of momenta,
Hamiltonian is unbounded !!

According to Ostrogradsky arguments and the lesson from second order derivative system, we expect that we have only to remove this linear momentum dependence for a healthy theory.

Conditions for a healthy theory

$$H = H_0 + P_{R^n} Q^n + \pi_{\psi^n} R^n.$$

- **First DC :** $a_{nm} - \alpha_{ni} A^{ij} \alpha_{mj} = 0.$ **(kinetic matrix is degenerate)**

  Assume det Aij ≠ 0 ⇒ $\Psi_n \equiv P_{Q^n} - e_{nm} Q^m - A^{ij} \alpha_{nj} \tilde{p}_i \approx 0.$ **(primary constraint)**

- **Second DC :** $\{\Psi_n, \Psi_m\} = -2[e_{nm} + \alpha_{ni}(A^{-1}CA^{-1})^{ij}\alpha_{mj}] = 0.$

  ⇒ $\Upsilon_n \equiv -\{\Psi_n, H\} = P_{R^n} - b_{nm} Q^m + \cdots \approx 0,$ **(secondary CS)**

- **Third DC :** $\{\Upsilon_n, \Psi_m\} = -b_{nm} - \alpha_{ni}[(4\bar{C}^2 + \bar{B})A^{-1}]^{ij}\alpha_{mj} = 0.$

  ⇒ $\Lambda_n \equiv -\{\Upsilon_n, H\} = \pi_{\psi^n} + 2 f_{nm} Q^m + \cdots \approx 0,$ **(tertiary CS)**

Now, the linear dependences are completely eliminated !!
No ghost (no instability) ???

The ghosts still lurk in the Hamiltonian

After erasing $P_{R^n}, b_{nm}, \pi_{\psi^n}$ using DCs and constraints, the Hamiltonian H reduces to

$$
\begin{aligned}
H = {} & \frac{1}{2}A^{ij}\bar{p}_i\bar{p}_j - \frac{1}{2}B_{ij}\bar{q}^i\bar{q}^j + \frac{1}{2}c_{nm}R^nR^m - \frac{1}{2}d_{nm}\psi^n\psi^m \\
& + \alpha_{ni}[(4\bar{C}^2 + \bar{B})A^{-1}]^{ij}\bar{p}_jR^n - 2\alpha_{ni}(\bar{C}\bar{B})^i{}_j\bar{q}^jR^n \\
& - 2[f_{nm} + 4\alpha_{ni}(\bar{C}^3A^{-1})^{ij}\alpha_{mj}]Q^mR^n,
\end{aligned}
$$

$$
(\bar{p}_i \equiv p_i - C_{ik}q^k + 2\alpha_{nk}A^{kl}C_{li}Q^n, \quad \bar{q}^i \equiv q^i + \alpha_{nk}A^{ki}Q^n)
$$

Q^n (= $\ddot{\psi}^n$) appears only linearly in H, making the Hamiltonian H unbounded !!

Eliminating the linear momentum terms is always necessary to kill the Ostrogradsky ghosts, but is not sufficient for higher-than-second-order derivative system.

Needs to fix Q^n as well.

Healthy theory

Canonical variables : $(\cancel{Q}^n, R^n, \psi^n, q^i, \cancel{\chi}_n, \cancel{\xi}_n)$ **primary constraints**

$\ddot{\psi}^n \quad \dot{\psi}^n$

Canonical momenta : $(P_{\cancel{Q}^n}, P_{\cancel{R}^n}, \pi_{\cancel{\psi}^n}, p^i, \rho_{\cancel{\chi}_n}, \rho_{\cancel{\xi}_n})$

$$H = H_0 + P_{R^n} Q^n + \pi_{\psi^n} R^n .$$

secondary & tertiary CS

- **Fourth DC :** $\{\Lambda_n, \Psi_m\} = 2(f_{nm} - \alpha_{ni} M^{ij} \alpha_{mj}) = 0.$

$\Rightarrow$ $\Omega_n \equiv -\{\Lambda_n, H\} = c_{nm} Q^m - d_{nm} \psi^m + \cdots \approx 0,$ **(quaternary CS)**

Q^n is now fixed (and expressed in terms of other variables).

$\Rightarrow$ **Only healthy DOFs** $(\psi^n, R^n(= \dot{\psi}^n), q^i, p^i)$ **remain.**

Hamiltonian is bounded and no ghosts.

$\Rightarrow$ **EOMs can be reduced to the second order system**

Dirac matrix

$$
D = \begin{array}{c|cccccc}
 & \multicolumn{3}{c}{\text{C1}} & \text{C4} & \text{C2} & \text{C3} \\
 & \Phi_\beta & \bar{\Phi}_\beta & \Psi_m & \Omega_m & \Upsilon_m & \Lambda_m \\
\hline
\Phi_\alpha & 0 & -\mathbf{1} & * & * & * & * \\
\bar{\Phi}_\alpha & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
\Psi_n & * & 0 & 0 & -Z_{mn} & 0 & 0 \\
\Omega_n & * & 0 & Z_{nm} & * & * & * \\
\Upsilon_n & * & 0 & 0 & * & 0 & Z_{mn} \\
\Lambda_n & * & 0 & 0 & * & -Z_{nm} & *
\end{array}
$$

Condition to complete Dirac procedure :

$$\det Z_{nm} = \det\{\Omega_n, \Psi_m\} \neq 0$$

$\Rightarrow$ **det D $\neq 0$** $\Rightarrow$ **All constraints are second class**

$\Rightarrow$ **Healthy (N+I) DOFs.**

Summary

- We have investigated how to obtain healthy degenerate theory with higher-than-second derivative.
- Eliminating linear momentum terms in the Hamiltonian is necessary and sufficient to kill the ghosts for second order derivative system.
- On the other hand, this is necessary but not sufficient for the Lagrangian with higher than second-order derivatives. The (lurked) linear dependence of canonical variables corresponding to second (even higher) derivatives must be removed as well.